On Nonlinear stage of the Modulational Instability. Part 1

V.E.Zakharov[a,b,c], A.A.Gelash[b]

[a]*Department of Mathematics, University of Arizona, Tucson, AZ, 857201, USA*
[b]*Novosibirsk State University, Pirogova, 2, Novosibirsk, 630090, Russia*
[c]*Physical Institute of RAS, Leninskiy prospekt, 53, Moscow, 119991, Russia*

Abstract

We describe a general N-solitonic solution of the focusing NLSE in presence of condensate by the use of dressing method. We separate special designated class of "regular solitonic" solutions do not disturbing phases of the condensate at infinity. All regular solitonic solutions can be treated as localized perturbations of the condensate. We found an important class of "regular" solutions which is a small perturbation at certain moment of time. They describe the nonlinear stage of the modulational instability of condensate.

Keywords: nonlinear schrödinger equation, modulational instability, soliton, dressing method

1. Introduction

This research was motivated by intention to develop an analytic theory of freak (or rogue) waves in ocean and optic fibers. In the recent time the simplest and most universal model for description of these waves is the focusing NLSE (Nonlinear Schrödinger Equation). In application to the theory of ocean waves this equation is used since 1968 [1]. In nonlinear optics it was known even earlier [2].

The focusing NLSE is the model of the first approximation. For the surface of fluid this model describes the essentially weakly nonlinear quasimonochromatic wave trains with maximal steepness not more than 0.15 only [3]. In nonlinear optics its application is also limited to the case of waves of small amplitudes (see, for instance [4]). Nowadays a lot of models generalizing the NLSE are developed. For the surface waves they are Dysthe equations [5, 6], for the waves in optic fibers equations include the third time derivatives and more complex forms of nonlinearity (see for instance [7, 8]). Also, freak waves in the ocean were studied by numerical modeling of exact Euler equations for potential flow with free boundary [9, 10]. The behavior of freak waves studied by NLSE and by more sophisticated models shows considerable quantitative difference. Nevertheless, advanced improvement of NLSE does not lead to any qualitatively new effects. That means that a careful and detailed study of NLSE solutions is still very important problem.

There is a point of common agreement that extreme waves on deep water appear as a result of modulational instability of quasimonochromatic weaky nonlinear stationarity waves [9, 11, 12]. In terms of the NLSE model it means that we should study instability of the "condensate" in the focusing version of this equation.

It is known since 1971 that the NLSE is a system completely integrable by the Inverse Scattering Method (ISM) [13]. Since this time hundreds of papers and several books have been written to this subject (see for instance the monographs [14, 15, 16]. In this sea of literature one can find some articles devoted to development of the ISM for NLSE with focusing in presence of condensate (see, for instance. This ISM technique in application NLSE not is novelty.

However the central question appearing in this theory - what is long-time consequence of the modulational instability it is not answered so far. In this article we will study only evolution of a special class of localized

initial data presented by exact n-solitonic solutions of NLSE. Solitonic solution in presence of unstable condensate have a long history.

In the nonsolitonic case a solution of auxillary linear Zakharov-Shabat system (wave function) is analytic in the right half-plane of spectral parameter $\lambda = \lambda_R + i\lambda_I$ with exeption of a cut on the real axis $0 < \lambda_R < A$, where A is the condensate amplitude. Each soliton adds a simple pole to some point of the right half plane including the cut.

The simplest solution was found by E.Kuznetsov in 1977 [17].Later on this solution was rediscovered by other authors [19, 18]. In this case the cut is posed on real axis outside the cut $\lambda_R > A$. The Kuznetsov solution is a localized bump oscillating in time. A period of oscillation growth as as $\lambda_R \to A$ and become infinite in the limiting case $\lambda_R = A$. In this limit Kuznetsov's solution turns to the instanton found in 1983 by Peregrine [20]. This is a pure homoclinic solution. It starts at $t \to -\infty$ from the condensate and return at $t \to +\infty$ to condensate with the same phase. The importance of these solutions for the development of the theory of freak waves is stressed in the article [25]. Later on multiinstanton solutions were found [23, 24].

In 1986 N. Akhmediev and V. Korneev discovered solitonic solution periodic in space but localized in time (Akhmediev's soliton) [27]. This solution is "almost homoclinic". It starts from a condensate and return to condensate with the same amplitude and different phase. In the Akhmediev case the pole is located inside the cut $0 < \lambda_R < A$.

In a general case the pole is located in any point on the right half-plane. A generic solutions moves and oscillates. At $x \to \pm\infty$ it goes to two condensates with the same amplitude and different phases. This solution was firstly found by M. Tajiri and Y. Watanabe in terms of Hirota's method. They also obtained general two-solitonic solution and suggested N-solitonic solution expression in 1998 [28]. However, in this article the properties of solitonic solutions except their stability were not studied. Later this solutions was obtained by some other methods and discussed in the frame of freak wave theory by A. Slunyaev at al. and N. Akhmediev at al. [30, 31, 33, 34]. In 2011 general one-solitonic solution was obtained by the authors of this article with the help of $\overline{\partial}$ problem [35].

In the recent time two-solitonic solutions and their degenerations were studied in the papers [28, 40] In this article we describe a general N-solitonic solution of the focusing NLSE in presence of condensate. We do not insist on our priority in this point but we believe that our method for its construction is the most simple and economic. But this is not a major point of our article. This article is more "practically oriented". We give a partial answer to a major question - what is the nonlinear stage of the modulational instability. When we speak about this subject we must clearly separate development of two types of initial perturbations. One can consider periodic or quasiperiodic in space perturbations. Or one can study development of perturbation localized in space. The time-behavior of these two types of perturbations is different even in the linear theory. The nonlinearity amplifiers this difference enormously. In this article we will speak only on nonlinear behavior of spatially localized perturbations. We will show that a certain class of small perturbations can be described by 2N - solitonic solutions. Mare exactly speaking - by 2N "near Akhmediev" solitons.

A general N-solitonic solution tends at $x \to \pm\infty$ to the condensate with opposite phases. In this article we separate special designated class of "regular solitonic" solutions do not disturbing phases of the condensate at infinity. All regular solitonic solutions can be treated as localized perturbations of the condensate. In general case they never are "small". However we are able to construct an ample class of regular solitonic solutions which is a certain moment of time (suppose at $t = 0$) are small perturbation of condensate. In fact they are pairs of "near-Akhmediev" solutions. In this part of our article we study only the simplest case of 2-solitonic solutions belonging to this remarkable class. In the second part of the paper we discuss the general situation.

2. NLSE via dressing method

We study solutions of the following NLSE

$$i\varphi_t - \frac{1}{2}\varphi_{xx} - (|\varphi|^2 - |A|^2)\varphi = 0 \tag{1}$$

2

with nonvanishing boundary conditions $|\varphi|^2 \to |A|^2$ at $x \to \pm\infty$. Without loss of generality we assume A is a real constant. Equation (1) is the compatibility condition for the following overdetermined linear system for a matrix function Ψ [13]:

$$\frac{\partial \Psi}{\partial x} = \widehat{U}\Psi \tag{2}$$

$$i\frac{\partial \Psi}{\partial t} = (\lambda\widehat{U} + \widehat{W})\Psi \tag{3}$$

Here

$$\widehat{U} = I\lambda + u, \qquad \widehat{W} = \frac{1}{2}\begin{pmatrix} |\varphi|^2 - A^2 & \varphi_x \\ \varphi_x^* & -|\varphi|^2 + A^2 \end{pmatrix} \tag{4}$$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad u = \begin{pmatrix} 0 & \varphi \\ -\varphi^* & 0 \end{pmatrix} \tag{5}$$

From (2), (3) one get

$$\frac{\partial \Psi^{-1}}{\partial x} = -\Psi^{-1}\widehat{U}$$

$$i\frac{\partial \Psi^{-1}}{\partial t} = -\Psi^{-1}(\lambda\widehat{U} + \widehat{W}) \tag{6}$$

and

$$\frac{\partial \Psi^+}{\partial x} = \Psi^+\widehat{U}^+$$

$$i\frac{\partial \Psi^+}{\partial t} = -\Psi^+(\lambda^*\widehat{U}^+ + \widehat{W}^+) \tag{7}$$

We consider equations (7) in the point $\widetilde{\lambda} = -\lambda^*$. One can see that

$$\widehat{U}^+(-\lambda^*) = -\widehat{U}(\lambda)$$

$$\widehat{W}^+ = \widehat{W} \tag{8}$$

Hence systems (6), (7) coincide. It means that they have a class of solutions Ψ satisfying the constrain.

$$\Psi^+(-\lambda^*) = \Psi^{-1}(\lambda) \tag{9}$$

Thereafter we will assume that this condition (9) is satisfied.

The idea of the dressing method is following [36]. Suppose we know some solution φ_0 of the NLSE (1) together with a fundamental solution Ψ_0 as a matrix function on x, t, λ, satisfying condition of corresponding linear system.

$$\frac{\partial \Psi_0}{\partial x} = \widehat{U}_0\Psi_0$$

$$i\frac{\partial \Psi_0}{\partial t} = (\lambda\widehat{U}_0 + \widehat{W}_0)\Psi_0 \tag{10}$$

$\widehat{U}_0$ and $\widehat{W}_0$ are obtained by replacing $\varphi \to \varphi_0$ in (4), (5). Then we introduce "the dressing function".

$$\chi = \Psi\Psi_0^{-1} \tag{11}$$

3

(notice that Ψ is still unknown!). We demand that χ is regular at infinity

$$\chi(\lambda) \to E + \frac{\widetilde{\chi}}{\lambda} + \cdots \qquad \text{at } |\lambda| \to \infty$$

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{12}$$

Apparently

$$\chi^{+}(-\lambda^{*}) = \chi^{-1}(\lambda) \tag{13}$$

The dressing function χ satisfies to overdetermined system of linear equations. The first equation is

$$\frac{\partial \chi}{\partial x} = \widehat{U}\chi - \chi\widehat{U}_0 \tag{14}$$

and

$$\frac{\partial \chi^{-1}}{\partial x} = -\chi^{-1}\widehat{U} + \widehat{U}_0\chi^{-1} \tag{15}$$

Equation (15) can be rewritten as follow

$$\widehat{U} = -\chi(\frac{\partial}{\partial x} - \widehat{U}_0)\chi^{-1} \tag{16}$$

Now if choose dressing function χ such that $\widehat{U}$ defined from (16) has no singularities on the λ-plane we construct new solution of equation (2).

According to the Louiville theorem in this case function $\widehat{U}$ must completely be defined by its asymptotics at $\lambda \to \infty$. By plugging (7) to (12) we find "dressing formula".

$$u = u_0 + [\widetilde{\chi}, I] \tag{17}$$

or

$$\varphi = \varphi_0 - 2\widetilde{\chi}_{(12)} \tag{18}$$

So far we performed dressing equation (2). From equation (3) we can derive following relation

$$\lambda\widehat{U} + \widehat{W} = -\chi\left(i\frac{\partial}{\partial t} - \lambda\widehat{U}_0 - \widehat{W}_0\right)\chi^{-1} \tag{19}$$

If we demand now that $\widehat{U}, \widehat{W}$ has no singularities on all λ - plane, including infinity (another word, we require that $\widehat{W}$ does not depend on λ), we realize "dressing" of equation (3) either. After performing the dressing procedure and determination of χ function

$$\Psi = \chi\Psi_0 \tag{20}$$

satisfies equations (2), (3) where φ is given by (18). This is a new solution of equation (1).

There several methods for constriction of dressing function χ. The method based on the Rieman-Hilbert problem was described in article of Zakharov and Shabat published in 1979 [36] More advanced approach is based on the use of the local $\overline{\partial}$-problem. It will a subject of our next paper. In this article we use "dressing for poor" making possible to construct multisolitonic solutions of NLSE on an arbitrary background.

3. General N-solitonic solution

In this chapter we construct solutions of the NLSE following the method developed by V. Zakharov and A. Mikhailov [38]. Let us assume that $\Psi_0(x, t, \lambda)$ is known and will consider that χ is a rational function of λ.

$$\chi = E + \sum_n \frac{U_n}{\lambda - \lambda_n} \tag{21}$$

Without loss of generality we can assume $Re\lambda_n > 0$. As χ satisfy the condition (13), χ^{-1} can be written as:

$$\chi^{-1} = E - \sum_n \frac{U_n^+}{\lambda + \lambda_n^*} \tag{22}$$

Let us denote

$$\chi_n = \chi\Big|_{\lambda = -\lambda_n^*} = E - \sum \frac{U_m}{\lambda_m^* + \lambda_n} \tag{23}$$

and consider identity

$$\chi\chi^{-1} = E \tag{24}$$

in the point $\lambda = -\lambda_n^*$ identity (24) means that

$$\chi_n U_n^+ = 0 \tag{25}$$

In the same way

$$\chi_n^+ U_n = 0 \tag{26}$$

Hence U_n, U_n^+ are degenerated matrixes, χ_n, χ_n^+ are degenerated either. Now we introduce two set of complex valued vectors $p_{n\alpha}$, $q_{n\alpha}$ and put

$$U_{n\alpha\beta} = p_{n\alpha}q_{n\beta}, \qquad U_{n\alpha\beta}^+ = q_{n\alpha}^* p_{n\beta}^*$$

Condition (25) means that

$$\chi_n q_{n\alpha}^* = 0 \tag{27}$$

$$\chi_{\alpha\beta} = \delta_{\alpha\beta} + \sum_n \frac{p_{n,\alpha}q_{n,\beta}}{\lambda - \lambda_n}$$

$$\chi_{\alpha\beta}^{-1} = \delta_{\alpha\beta} - \sum_n \frac{q_{n,\alpha}^* p_{n,\beta}^*}{\lambda + \lambda_n^*} \tag{28}$$

Let us plug (28) to equation (16). In a general case function U acquires poles at points $\lambda = \lambda_n$ and $\lambda = -\lambda_n^*$. To perform dressing we must annulate residues in all poles. We consider equation (16) in neighborhood of the point $\lambda = -\lambda_n^*$. To kill the residue in this point we have to demand

$$\chi_n\left(\frac{\partial}{\partial x} - \widehat{U}_0(-\lambda_n^*)\right)q_{n\alpha}^* p_{n\beta}^* = 0 \tag{29}$$

or

$$\chi_n\left(\frac{\partial q_n^*}{\partial x} - U_0(-\lambda_n^*)q_{n\alpha}^* + q_{n\alpha}^*\frac{\partial p_{n\beta}^*}{\partial x}\right) = 0 \tag{30}$$

However in virtue of (27) the last term in (30) is cancelled. Hence it is enough to demand

$$\frac{\partial q_{n\alpha}^*}{\partial x} - \widehat{U}_0(-\lambda_n^*)_{\alpha\beta}q_{n\beta}^* = 0 \tag{31}$$

5

This equality is resolved as follow

$$q^*_{n\alpha} = \Psi_{0,\alpha\beta}(-\lambda^*_n)\xi_{n\beta} \tag{32}$$

Here ξ_α is arbitrary complex-valued vector. Thereafter we will use notation

$$F_{n\alpha\beta} = \Psi_{0,\alpha\beta}(-\lambda^*_n) \tag{33}$$

Since this moment we consider that the set of vectors $\overrightarrow{q}_n$, $\overrightarrow{q}^*_n$ is known. To find the second $\overrightarrow{p}_n$, $\overrightarrow{p}^*_n$ we must resolve relation (27) which is equivalent to the system of linear algebraic equations.

$$\sum_m \frac{(\overrightarrow{q}_n \cdot \overrightarrow{q}^*_m)}{\lambda_n + \lambda^*_m} \overrightarrow{p}^*_m = \overrightarrow{q}_n \tag{34}$$

Here $(\overrightarrow{q}_n \cdot \overrightarrow{q}^*_m) = q_{n,1}q^*_{m,1} + q_{n,2}q^*_{m,2}$ is a scalar product of $\overrightarrow{q}_n$ and $\overrightarrow{q}_m$ vectors. Let us denote:

$$M_{nm} = \frac{(\overrightarrow{q}_n \cdot \overrightarrow{q}^*_m)}{\lambda_n + \lambda^*_m}$$

and $M = det(M_{nm})$. M is a Hermitian matrix:

$$M^*_{nm} = M_{mn} = M^T_{nm} \tag{35}$$

Now (32) can be rewritten as

$$\sum_m M^T_{nm} \overrightarrow{p}_m = \overrightarrow{q}^*_n \tag{36}$$

We need to find $\widetilde{\chi}$ from asymptotic expansion of χ (12), which can be represented as:

$$\widetilde{\chi}_{\alpha\beta} = \sum_n p_{n,\alpha}q_{n,\beta}$$

This sum can be calculated as determinant ratio:

$$\widetilde{\chi}_{\alpha\beta} = \frac{\widetilde{M}_{\alpha\beta}}{M} \tag{37}$$

Here $\widetilde{M}_{\alpha\beta}$ is extended matrix:

$$\widetilde{M}_{\alpha\beta} = \begin{vmatrix} 0 & q_{1,\beta} & \cdots & q_{n,\beta} \\ q^*_{1,\alpha} & & & \\ \vdots & & M^T_{nm} & \\ q^*_{n,\alpha} & & & \end{vmatrix} \tag{38}$$

Here M^T - transposed matrix. We will find the the solution of NLSE (1) from condition (18) as:

$$\varphi = \varphi_0 - 2\frac{\widetilde{M}_{12}}{M} \tag{39}$$

Notice that formula (37)(38) in the simplest case dressing on zero background were found by Faddeev and Takhtajan [14].

Notice also that function $\chi_{\alpha\beta}$ also can be presented as ratio of two determinants.

$$\chi_{\alpha\beta} = \frac{\widehat{M}_{\alpha\beta}}{M} \tag{40}$$

Here

$$\widehat{M}_{\alpha\beta} = \begin{vmatrix} \delta_{\alpha\beta} & \frac{q_{1,\beta}}{\lambda-\lambda_1} & \cdots & \frac{q_{n,\beta}}{\lambda-\lambda_n} \\ q_{1,\alpha}^* & & & \\ \vdots & & M_{nm}^T & \\ q_{n,\alpha}^* & & & \end{vmatrix} \tag{41}$$

The dressing formula (18) can be written in the explicit form. Let

$$\Psi_0 = \begin{pmatrix} \Psi_{011} & \Psi_{012} \\ \Psi_{021} & \Psi_{022} \end{pmatrix} \tag{42}$$

Then

$$\Psi = \chi \frac{1}{M} \begin{pmatrix} \widehat{M}_{11}\Psi_{011} + \widehat{M}_{12}\Psi_{021} & \widehat{M}_{11}\Psi_{012} + \widehat{M}_{12}\Psi_{022} \\ \widehat{M}_{21}\Psi_{011} + \widehat{M}_{22}\Psi_{021} & \widehat{M}_{21}\Psi_{012} + \widehat{M}_{22}\Psi_{022} \end{pmatrix} \tag{43}$$

Now mention that transformation

$$q_n \to a_n q_n \qquad p_n \to \frac{1}{a_n} p_n \tag{44}$$

where a_n are arbitrary complex numbers does not change the result of dressing. Thus one can put

$$\xi_n = \begin{pmatrix} 1 \\ C_n \end{pmatrix} \tag{45}$$

Finally constructed n-solitonic solution depends on $2n$ sets of complex numbers λ_n, C_n or of $4n$ real parameters. Thereafter we will assume $Re\lambda > 0$ In fact, it is enough to enlist all possible solitonic solutions. This fact will be proven in a separate paper. So far we are sure that $\lambda_n + \lambda_m^* \neq 0$, thus equations (34) always are resolvable.

Now we present one-solitonic solution. Function χ has only one pole at $\lambda = \eta$, while χ^{-1} has a pole at $\lambda = -\eta^*$. They can be presented in the following form (see [13]).

$$\chi = E + \frac{U}{\lambda - \eta}$$
$$\chi^{-1} = E - \frac{U^+}{\lambda + \eta^*} \tag{46}$$

As before

$$U_{\alpha\beta} = p_\alpha q_\beta \tag{47}$$

Vectors p_α, q_α are connected now with relation

$$p_\alpha = \frac{(\eta + \eta^*)q_\alpha^*}{|q|^2} \tag{48}$$

As a result χ, χ^{-1} are

$$\chi = E + \frac{(\eta + \eta^*)P}{\lambda - \eta}, \qquad \chi^{-1} = E - \frac{(\eta + \eta^*)P^+}{\lambda + \eta^*} \tag{49}$$

Here

$$P_{\alpha\beta} = \frac{q_\alpha^* q_\beta}{|q_1|^2 + |q_2|^2} \tag{50}$$

$P^2 = P$ thus P is projective matrix. As before we consider that the "seed" matrix $\Psi_0(x,t,\lambda)$ is known. According to our defenition (33)

$$F_{\alpha\beta} = \Psi_{0,\alpha\beta}(-\eta^*) \tag{51}$$

7

now

$$q_1^* = F_{11} + CF_{12}$$
$$q_2^* = F_{21} + CF_{22} \tag{52}$$

C is arbitrary complex constant. Finally a new solution of NLSE is

$$\varphi = \varphi_0 - \frac{2(\eta + \eta^*)q_1^*q_2}{|q_1|^2 + |q_2|^2} \tag{53}$$

This formula presents the solitonic solution on an arbitrary background. It was established first in 1979 by Zakharov and Shabat [36] and reobtained in 1988 by Its, Rybin and Sall [39] (see also [41]).

All results of this chapter can be extended to much more general class of nonlinear wave systems which can be presented as compatibility condition for overdetermined linear system

$$\Psi_x = U(\lambda)\Psi$$
$$\Psi_t = V(\lambda)\Psi \tag{54}$$

Where U, V are rational matrix 2×2 functions on λ satisfying the follows condition

$$U^+(-\lambda^*) = -U(\lambda)$$
$$V^+(-\lambda^*) = -V(\lambda) \tag{55}$$

In particulary it is extended to all higher members of the NLSE hierarchy. One need only minor generalization to extend this procedure to the case of $n \times n$ matrix systems.

4. N-solitonic solution on condensate

Since this moment we study dressing only on the condensate background. Now one can put $\varphi_0 = A$.

$$U_0 = \begin{pmatrix} \lambda & A \\ -A & -\lambda \end{pmatrix} \qquad \widehat{W}_0 = 0 \tag{56}$$

And Ψ_0 can be found as

$$\Psi_0(x, t, \lambda) = \frac{1}{\sqrt{1 - s^2(\lambda)}} \begin{pmatrix} e^{\phi(x,t,\lambda)} & s(\lambda) \cdot e^{-\phi(x,t,\lambda)} \\ s(\lambda) \cdot e^{\phi(x,t,\lambda)} & e^{-\phi(x,t,\lambda)} \end{pmatrix} \tag{57}$$

Here

$$\phi = kx + \Omega t, \ \ k^2 = \lambda^2 - A^2, \ \ \Omega = -i\lambda k, \ \ s = -\frac{A}{\lambda + k}$$

Thereafter we assume that function $k(\lambda) = \sqrt{\lambda^2 - A^2}$ has a cut $-A < Re\lambda < A$. Thus $k(\lambda) \to \lambda$ at $\lambda \to \infty$. Then

$$\Psi_0^{-1}(x, t, \lambda) = \frac{1}{\sqrt{1 - s^2(\lambda)}} \begin{pmatrix} e^{-\phi(x,t,\lambda)} & -s(\lambda) \cdot e^{-\phi(x,t,\lambda)} \\ -s(\lambda) \cdot e^{\phi(x,t,\lambda)} & e^{\phi(x,t,\lambda)} \end{pmatrix} \tag{58}$$

Notice that

$$k^*(-\lambda^*) = -k(\lambda), \quad s^*(-\lambda^*) = -s(\lambda), \quad \phi^*(-\lambda^*) = -\phi(\lambda)$$

One can check that

$$\Psi_0^{-1}(-\lambda^*) = \Psi_0^+(\lambda) \tag{59}$$

8

Thereafter we denote for simplicity.

$$\phi_n = \phi_n(\lambda_n) \qquad s_n = s(\lambda_n) \tag{60}$$

and by virtue of (59)

$$\phi_n(-\lambda_n^*) = -\phi_n^* \qquad s_n(-\lambda_n^*) = -s_n^* \tag{61}$$

Then

$$F_n = \Psi_0(-\lambda_n^*) = \begin{pmatrix} e^{-\phi_n^*} & -s_n^* \cdot e^{\phi_n^*} \\ -s_n^* \cdot e^{-\phi_n^*} & e^{\phi_n^*} \end{pmatrix} \qquad q_n^* = F_n \begin{pmatrix} 1 \\ C_n \end{pmatrix} \tag{62}$$

(we can omit factor $\frac{1}{\sqrt{1-s_n^2}}$ because, as we mention before, it does not change the result of dressing)and

$$q_{n1} = e^{-\phi_n} - C_n^* s_n e^{\phi_n} \qquad q_{n2} = -s_n e^{-\phi_n} + C_n^* e^{\phi_n} \tag{63}$$

So far we assumed that χ is a rational function on the λ-plane with cut $-A < Re\lambda < A$. Now we perform the Jukowsky transform and map this plane onto the outer part of the circle of unit radius (see Fig. 1.).

$$\lambda = \frac{A}{2}(\xi + \frac{1}{\xi}) \qquad k = \frac{A}{2}(\xi - \frac{1}{\xi}) \qquad s = -\frac{1}{\xi} \qquad \xi + \xi^* \neq 0 \tag{64}$$

The λ-plane is mapped onto exterior of the circle of radius R. If the pole is posed at $\lambda = \lambda_n$, $\lambda_n = \frac{A}{2}(\xi_n + \frac{1}{\xi_n})$.
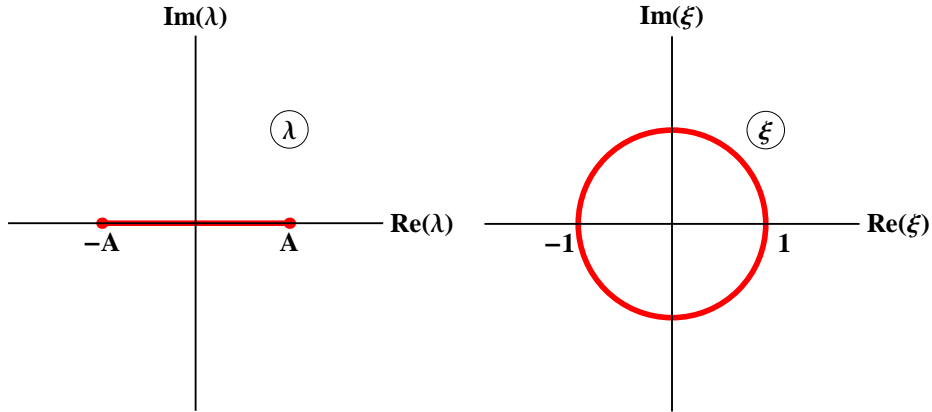
Figure 1: Uniformization of the plane of spectral parameter with the help of Jukovsky transform.

In new variables

$$F_n = \begin{pmatrix} e^{-\phi_n^*} & \frac{1}{\xi_n^*} \cdot e^{\phi_n^*} \\ \frac{1}{\xi_n^*} \cdot e^{-\phi_n^*} & e^{\phi_n^*} \end{pmatrix} \tag{65}$$

$$q_{n1} = e^{-\phi_n} + \frac{1}{\xi_n} C_n^* e^{\phi_n} \qquad q_{n2} = \frac{1}{\xi_n} e^{-\phi_n} + C_n^* e^{\phi_n} \tag{66}$$

Finally we use parametrization

$$\xi_n = R_n e^{i\alpha_n} \qquad C_n = e^{i\theta_n + \mu_n} \tag{67}$$

9

and thereafter we denote $R_n = e^{z_n}$. Now

$$\lambda_n = \frac{A}{2}\left(R_n + \frac{1}{R_n}\right)\cos(\alpha_n) + \frac{iA}{2}\left(R_n - \frac{1}{R_n}\right)\sin(\alpha_n) =$$
$$A\left[\cosh(z_n)\cos(\alpha_n) + i\sinh(z_n)\sin(\alpha_n)\right] \tag{68}$$

and we also denote

$$w_n = -i\alpha_n - z_n \tag{69}$$

Now

$$F_n = \begin{pmatrix} e^{-\phi_n^*} & e^{w_n^* + \phi_n^*} \\ e^{w_n^* - \phi_n^*} & e^{\phi_n^*} \end{pmatrix} \tag{70}$$

Expressions for $\overrightarrow{q}_n$ can be simplified by redefinition of phase factor ϕ_n. One can put

$$q_{n1} = e^{-\phi_n} + e^{w_n + \phi_n} \qquad\qquad q_{n2} = e^{w_n - \phi_n} + e^{\phi_n} \tag{71}$$

Now

$$\phi_n = u_n + iv_n \tag{72}$$

Here

$$u_n = \text{æ}_n x - \gamma_n t + \frac{1}{2}\mu_n \qquad v_n = k_n x - \omega_n t - \frac{1}{2}\theta_n \tag{73}$$

Here

$$\text{æ}_n = \frac{A}{2}\left(R_n - \frac{1}{R_n}\right)\cos(\alpha_n) = A\sinh(z_n)\cos(\alpha_n)$$
$$k_n = \frac{A}{2}\left(R_n + \frac{1}{R_n}\right)\sin(\alpha_n) = A\cosh(z_n)\sin(\alpha_n)$$
$$\gamma_n = -\frac{A^2}{4}\left(R_n^2 + \frac{1}{R_n^2}\right)\sin(2\alpha_n) = -\frac{A^2}{2}\cosh(2z_n)\sin(2\alpha_n)$$
$$\omega_n = \frac{A^2}{4}\left(R_n^2 - \frac{1}{R_n^2}\right)\cos(2\alpha_n) = \frac{A^2}{2}\sinh(2z_n)\cos(2\alpha_n) \tag{74}$$

The N - solitonic solution is invariant with respect to shifts in time and space. If one replace

$$x \to x + x_0 \qquad t \to t + t_0 \tag{75}$$

then

$$\mu_n \to \mu_n + 2(\text{æ}_n x_0 + \gamma_n t_0) \qquad \theta_n \to \theta_n + 2(k_n x_0 - \omega_n t_0) \tag{76}$$

It means that a space-time shift leads to renormalization of constants C_n.

$$C_n \to C_n e^{2(\text{æ}_n x_0 + \gamma_n t_0) + 2i(k_n x_0 - \omega_n t_0)} \tag{77}$$

N-soliton solution can be considered as a nonlinear superposition of n separate solitons. Each of them is characterized by the group velocity.

$$V_{Gr_n} = -\frac{\gamma_n}{\text{æ}_n} \tag{78}$$

and phase velocity

$$V_{Ph_n} = -\frac{\omega_n}{k_n} \tag{79}$$

In the case when all group velocities coincide the solution presents the "solitonic atom" first mentioned in [36] - a complicated configuration of solitons moving together. Complicated solitonic atoms containing a large amount of solitons should be described by method of statistical mechanics. In two-solitonic case

$$\frac{R_1^4 + 1}{R_{1,2}(R_1^2 - 1)}\sin(\alpha_1) = \frac{R_2^4 + 1}{R_2(R_2^2 - 1)}\sin(\alpha_2) \tag{80}$$

Shapes of a typical "two-solitonic atom" and "three-solitonic atom" is presented on Fig. 2 and Fig. 3
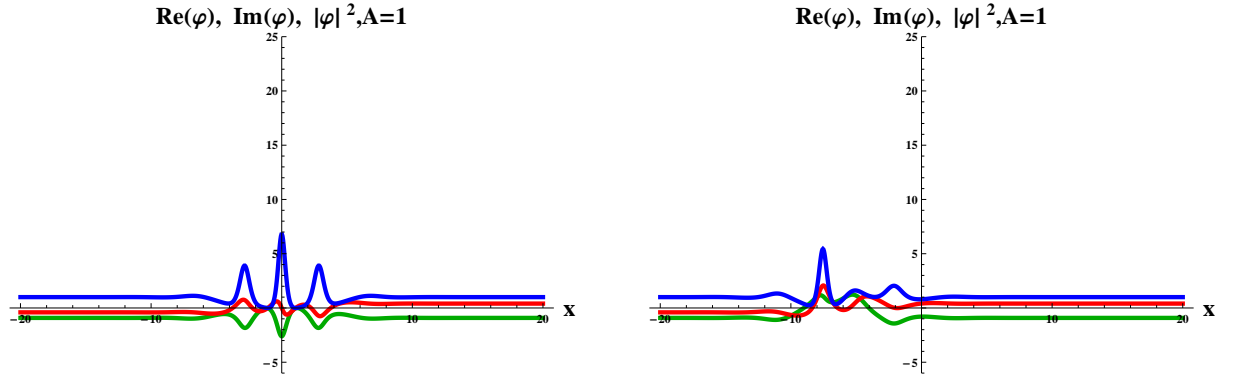
Figure 2: Two-solitonic atom at different moments of time. $R_1 = 1.5, R_2 = 2, \alpha_1 = \frac{\pi}{5}, \alpha_2 = 0.735242$ $Re(\varphi)$ - Green, $Im(\varphi)$ - Red, $|\varphi|^2$ - Blue
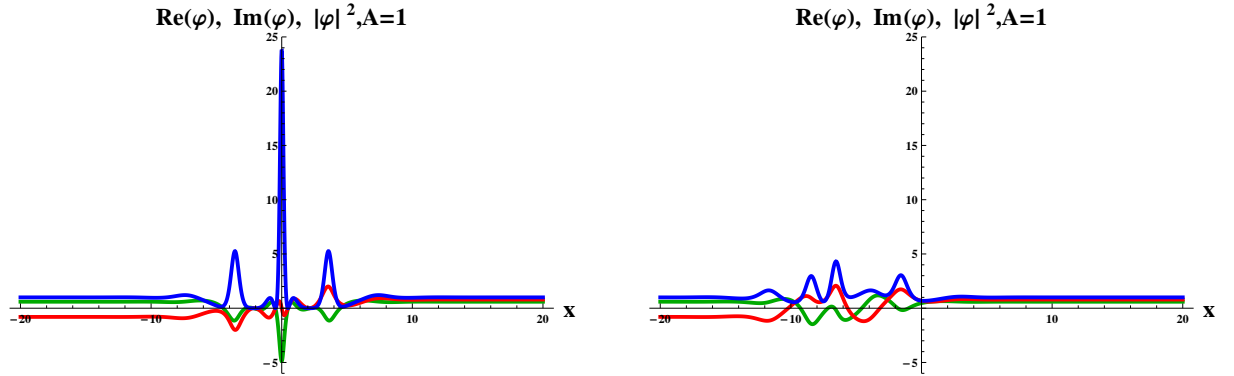
Figure 3: Three-solitonic atom at different moments of time. $R_1 = 1.5, R_2 = 2, R_3 = 2.5 \alpha_1 = \frac{\pi}{5}, \alpha_2 = 0.735242, \alpha_3 = 0.6721$ $Re(\varphi)$ - Green, $Im(\varphi)$ - Red, $|\varphi|^2$ - Blue

If all group velocities are different the N - solitonic solution asymptotically at $t \to \pm\infty$ separates to superposition of far remote from each other individual solitons. This fact makes possible to determine asymptotic properties of N-solitonic solution. We will discuss only the case when all $z_n > 0$, thus the solution not contain "Akhmediev components". Now at $x \to \pm\infty$

$$\varphi \to Ae^{\imath\alpha^\pm} \tag{81}$$

11

Apparently phases $\alpha^{\pm}$ are constants in time. For far separated solitons

$$\alpha^+ = 2(\alpha_1 + \cdots + \alpha_n) \qquad \alpha^- = -2(\alpha_1 + \cdots + \alpha_n) \tag{82}$$

This fact holds even in a general case when some group velocity could coincide.

We developed the analytical code making possible to calculate N-solitonic solutions by the use of the "Mathematica" software and checked relations (82) directly for $n = 2, 3$.

If we are interested by N-solitonic solutions localized in finite domain of space and not perturbing far remote condensate we must put

$$\alpha_1 + \cdots + \alpha_n = 0 \tag{83}$$

We call this solution "regular solitonic solution" and thereafter study only this case. If we assume that the modulation instability develops from localized perturbation, only regular solution can be used as model for its nonlinear behavior.

Among one-solitonic solutions only Kuznetsov-Ma and Peregrine solution are regular. In two-solitonic case we can construct a broad class of regular solutions.

5. One soliton solution on condensate

One soliton solution on condensate background can be obtained by implementation of the results of the last part of §3. This solution is defined only by one complex eigenvalue $\lambda_1 = \eta$. Let us denote

$$\phi = \phi(\eta) \qquad s = s(\eta) \tag{84}$$

remember that by virtue of (59)

$$\phi(-\eta^*) = -\phi^* \qquad s(-\eta^*) = -s^* \tag{85}$$

After uniformization

$$q_1 = e^{-\phi} - C^* s e^{\phi} \qquad q_2 = -s e^{-\phi} + C^* e^{\phi} \tag{86}$$

and the general one-solitonic solution is presented by expression.

$$\varphi = A - A(\xi + \xi^*) \frac{\frac{1}{\xi} + |C|^2 \frac{1}{\xi^*} e^{2(\phi+\phi^*)} + C^* e^{2\phi} + C \frac{1}{|\xi|^2} e^{2\phi^*}}{1 + |C|^2 e^{2(\phi+\phi^*)} + C^* \frac{\xi+\xi^*}{1+|\xi|^2} e^{2\phi} + C \frac{\xi+\xi^*}{1+|\xi|^2} e^{2\phi^*}} \tag{87}$$

After our standard parametrization:

$$\xi = R e^{i\alpha}, \qquad C = e^{i\theta + \mu}, \qquad R = e^z$$

we obtain (Thereafter we call R - amplitude and α - argument of a solution)

$$\phi = u + iv$$
$$u = \text{æ}x - \gamma t \qquad v = kx - \omega t$$
$$\text{æ} = \frac{A}{2}\left(R - \frac{1}{R}\right)\cos(\alpha), \quad \gamma = -\frac{A^2}{4}\left(R^2 + \frac{1}{R^2}\right)\sin(2\alpha)$$
$$k = \frac{A}{2}\left(R + \frac{1}{R}\right)\sin(\alpha), \quad \omega = \frac{A^2}{4}\left(R^2 - \frac{1}{R^2}\right)\cos(2\alpha)$$

The general one-solitonic solution depends on four scalar parameters R, α, θ, μ. Two of them θ, μ are responsible for shift in time and in space. Thereafter we denote $R = e^z$. Then

$$\text{æ} = A \sinh(z)\cos(\alpha), \quad \gamma = -\frac{A^2}{2}\cosh(2z)\sin(2\alpha)$$
$$k = A \cosh(z)\sin(\alpha), \quad \omega = \frac{A^2}{2}\sinh(2z)\cos(2\alpha)$$

12

If we put $\mu = 0, \theta = 0$, the one-solitonic solution can be written as follows

$$\varphi = -\frac{A}{\cosh(z)\cosh(2u) + \cos(\alpha)\cos(2v)} \times$$

$$\left[\cosh(z)\cos(2\alpha)\cosh(2u) + \cosh(2z)\cos(\alpha)\cos(2v) + i\left(\cosh(z)\sin(2\alpha)\sinh(2u) + \sinh(2z)\cos(\alpha)\sin(2v)\right)\right] \quad (88)$$

To obtain the general solution one must replace $t \to t - t_0$, $x \to x - x_0$ where $t_0 = \frac{1}{2}\frac{\mu}{\gamma}$, $x_0 = \frac{1}{2}\frac{\theta}{k}$.

This solution is localized in space if $R \neq 1$. In this case asymptotics of (88):

$$\varphi \to -Ae^{2i\alpha} \quad \text{at} \quad x \to -\infty$$
$$\varphi \to -Ae^{-2i\alpha} \quad \text{at} \quad x \to +\infty$$
$$|\varphi|^2 = A^2 \quad \text{at} \quad x \to \pm\infty$$

We see that solution (88) has identical asymptotics at $x \to \pm\infty$ only in the case $\alpha = 0$, when the pole is posed on the real axis.

The position of pole defines a type soliton. Different possible position are plotted in the Fig. 4 We
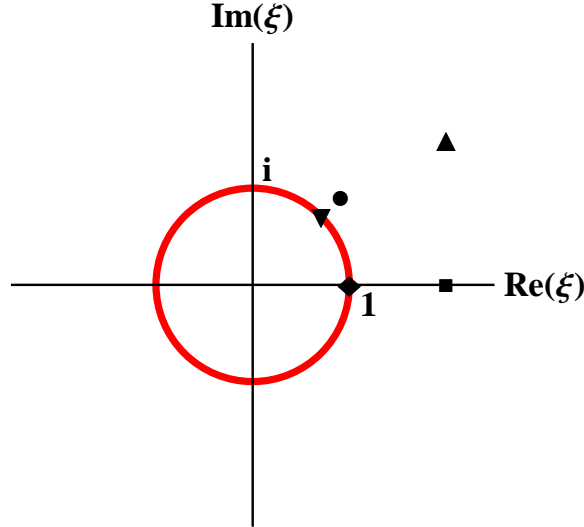
Figure 4:
"Species" of one-solitonic solution. ■ - Kuznetsov-Ma, ▼ - Akhmediev ▲ - General, ● -near Akhmediev, ♦ - Peregrine

describe below all these "species" of solitonic solution.

Let us put $\alpha = 0, R > 1, z > 0$. This is exactly Kuznetsov-Ma solution. If the solution is centered at $x = 0$ and $\theta = 0$ it is presented by following expression. Then

$$\varphi = -\frac{A}{\cosh(z)\cosh(2u) + \cos(2v)}\left[\cosh(z)\cosh(2u) + \cosh(2z)\cos(2v) + i\sinh(2z)\sin(2v)\right] \quad (89)$$

Here

$$u = A\sinh(z)x \qquad v = \frac{A^2}{2}\sinh(2z)t \quad (90)$$

This solution is periodic in time. Period of its oscillations

$$T = \frac{4\pi}{A\sinh(2z)} \quad (91)$$

13

$$T \to \infty \quad \text{if} \quad z \to 0, \ R \to 1$$
$$T \to 0 \quad \text{if} \quad z \to \infty, \ R \to \infty \tag{92}$$

In the limit $R \to 1, z \to 0$ Kuznetsov-Ma soliton turns to the standard standing soliton of NLSE on zero background. A typical behavior of Kuznetsov-Ma soliton is presented on Fig. 5. In the limit $R \to 1 \ T \to \infty$

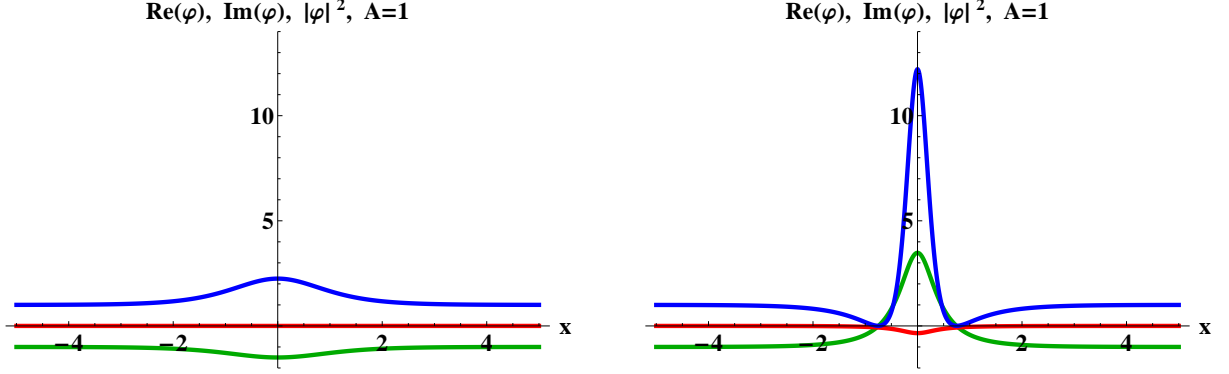Figure 5: Kuznetsov soliton at the moment of minimum(left) and maximum(right) of its amplitude. $R = 2$. $Re(\varphi)$ - Green, $Im(\varphi)$ - Red, $|\varphi|^2$ - Blue

the Kuznetsov solutions turns to the Peregrine homoclinic soliton presented by expression

$$\varphi = A\left(1 - 4\frac{1 - 2it}{1 + 4x^2 + 4t^2}\right) \tag{93}$$

This solution was found by Peregrine in 1983 [20] and reopened by Its, Rybin and Sall in 1988 [39]. In their article it was called "exulton". This solution is actually "instanton" appearing from the condensate and disappearing. Indeed $\varphi \to A$ at $t \to \pm\infty$. More general "multiinstantonic" solutions are found in papers [23, 24]. It is a popular idea to treat this solution as a model for freak waves [25, 26]. Multisolitonic solutions describe a special scenario of modulation instability development, when growing perturbation in the long run return back to condensate. We will show in this article that this is a very special scenario, badly unstable with respect to small deformations. In a more general case waves developed from small perturbation do exist infinitely long time. The Peregrine soliton is plotted on Fig. 6 It was reported that this solution was observed experimentally in water wave tank [22] and optical fibers [21].

Let us put now $R = 1, z = 0$. Again we will put $\theta = 0, \mu = 0$. We obtain famous "Akhmediev breather" periodic in space and localized in time.

$$\varphi = -A\left[\frac{\cos(2\alpha)\cosh(2u) + \cos(\alpha)\cos(2v) + i\sin(2\alpha)\sinh(2u)}{\cosh(2u) + \cos(\alpha)\cos(2v)}\right] \tag{94}$$

here

$$u = \frac{1}{2}A^2\sin(2\alpha)t \quad v = A\sin(\alpha)x$$

Akhmediev solution is homoclinic in a "weak sense". It has following asymptotic behavior at $t \to \pm\infty$.

$$\varphi \to -Ae^{-2i\alpha} \quad \text{at} \quad t \to -\infty$$
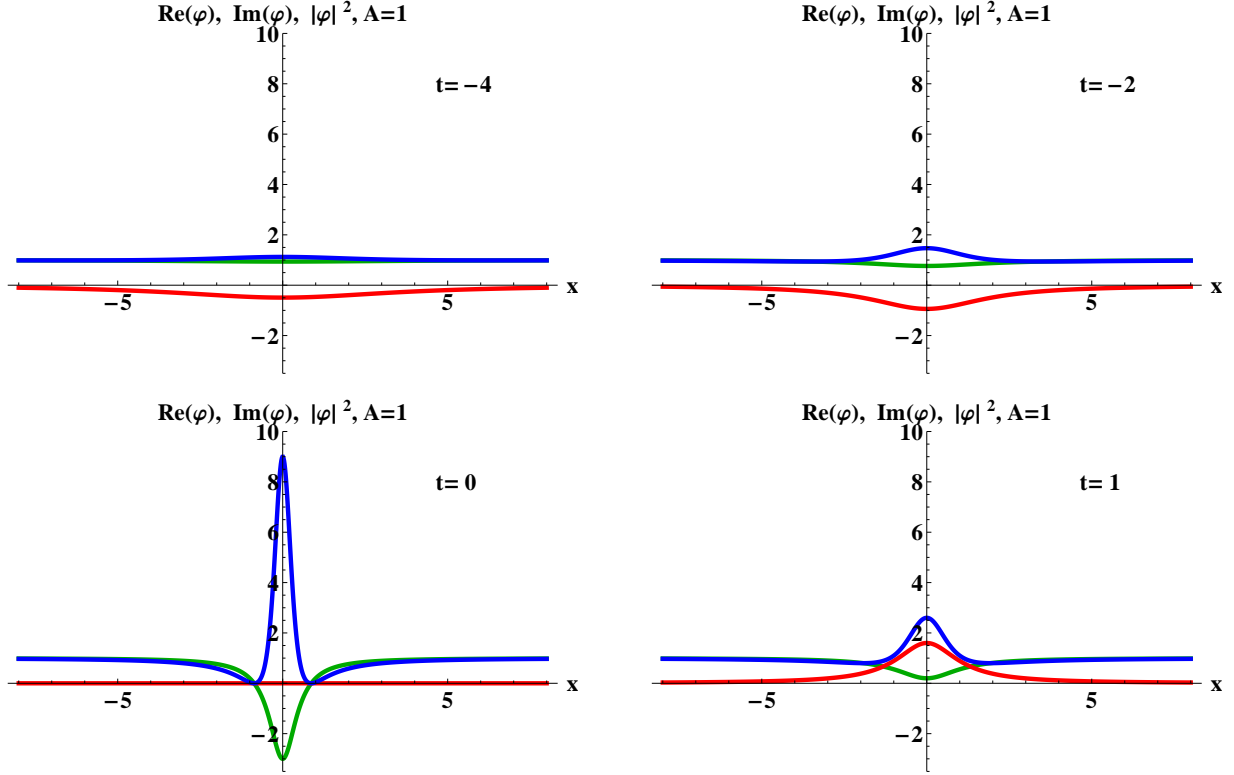$$\varphi \to -Ae^{2i\alpha} \quad \text{at} \quad t \to +\infty$$

14

Figure 6: Peregrin breather at different moments of time. $Re(\varphi)$ - Green, $Im(\varphi)$ - Red, $|\varphi|^2$ - Blue

Hence $|A^+|^2 = |A^-|^2$, but $A^+ \neq A^-$. In the especially interesting case $\alpha = \frac{\pi}{4}$ $A^+ = -A^-$. Spatial period of the Akhmediev breather

$$L = \frac{2\pi}{A \sin(\alpha)} \tag{95}$$

Tends to infinity if $\alpha \to 0$. In this limit solution tends to a periodic set of distantly remote Peregrine solitons. The Akhmediev breather is plotted on Fig. 7.

Now we return to the general one-solitonic solution. In the explicit form it was found in the paper of Tajiri and Watanabe [28] and reobtained then by several authors [29, 33, 30, 35]. This is a localized object of size $l \approx \frac{1}{A \cos(\alpha) \sinh(z)}$ propagating along the condensate with group velocity

$$V_{gr} = \frac{\gamma}{\ae} = A \frac{\cosh(2z)}{\sinh(z)} \sin(\alpha) \tag{96}$$

Soliton's amplitude oscillates with angular frequency ω. Soliton is filled with carrying wave propagating with phase velocity.

$$V_{ph} = 2A \sinh(z) \frac{\cos(2\alpha)}{\sin(\alpha)} \tag{97}$$

If $\alpha \to 0$ this carrying wave vanishes. A typical "general" soliton is plotted on Fig. 8.

It is important to average by integration over the period of oscillations $T = \frac{2\pi}{\omega}$ the squared absolute value of onesolitonic solution (88). It demonstrates the behavior of average value of the Hamiltonian. Remarkably that the answer can be obtained analytically in very simple form. This requires move to a coordinate system which travel with group velocity V_{gr}
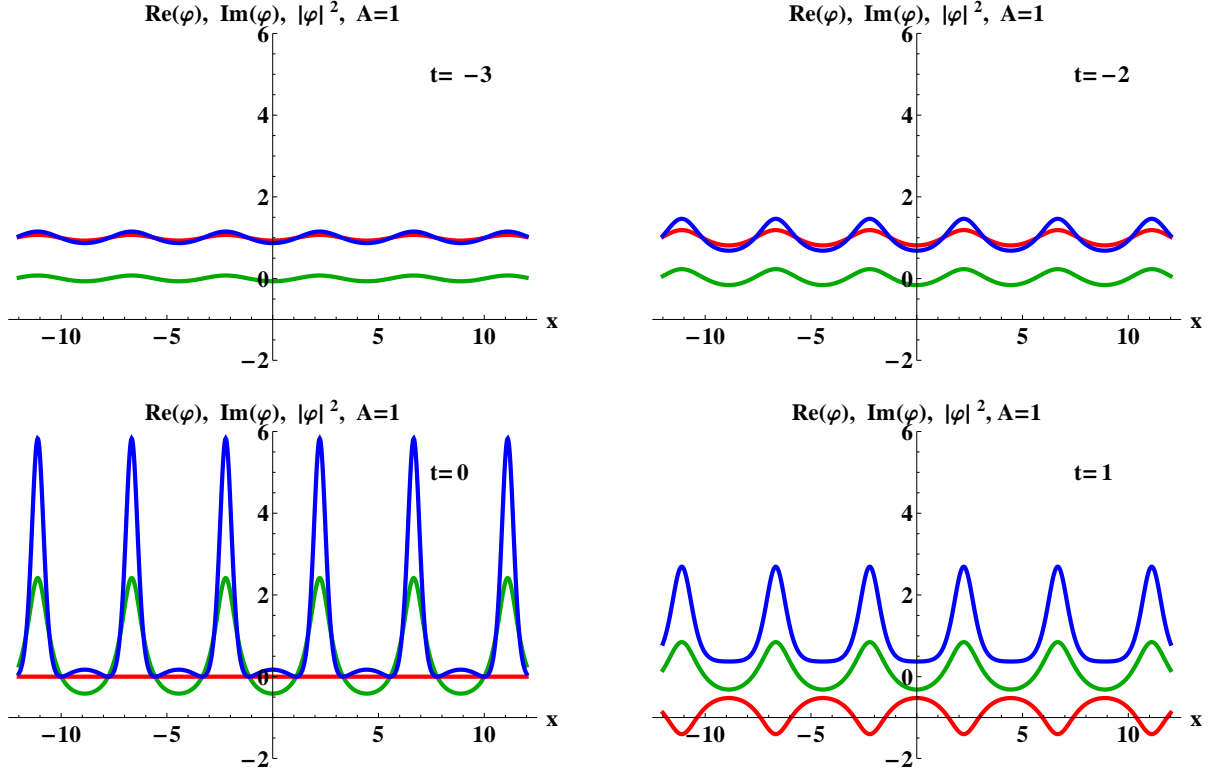
$$x \to x + \frac{\gamma}{\ae} t \tag{98}$$

15

Figure 7: Akhmediev breather at different moments of time. $R = 1, \alpha = \frac{\pi}{4}, \mu = 0$. $Re(\varphi)$ - Green, $Im(\varphi)$ - Red, $|\varphi|^2$ - Blue

Then the dependence on time of hyperbolic functions disappear and they can be regarded as constant when integrating over time. As a result

$$< |\varphi|^2 >_T = \frac{A^2}{2\pi} \int_0^{2\pi} \frac{1}{\left(\cosh(z)\cosh(2u) + \cos(\alpha)\cos(\tau)\right)^2} \times$$

$$\left[\left(\cosh(z)\cos(2\alpha)\cosh(2u) + \cosh(2z)\cos(\alpha)\cos(\tau)\right)^2\right.$$

$$\left. + \left(\cosh(z)\sin(2\alpha)\sinh(2u) + \sinh(2z)\cos(\alpha)\sin(\tau)\right)^2\right] d\tau \tag{99}$$

Integral is taken by residues after the standard change of variables $w = e^{i\tau}$. The path of integration now $|w| = 1$. The integrand has one pole of the first order at point $z_0 = 0$ and two poles of the second order in points $z_{1,2} = -\frac{\cosh(z)\cosh(2u)}{\cos(\alpha)} \pm \sqrt{\frac{\cosh^2(z)\cosh^2(2u)}{\cos^2(\alpha)} - 1}$. However, only w_0 and w_1 lie inside the circle $|w| = 1$. After integration we need back to initial coordinate system. The final answer is

$$< |\varphi|^2 >_T = A^2 \left(1 + \frac{4\cosh(2u)}{\left[\cosh^2(2u) - \frac{\cos^2(\alpha)}{\cosh^2(z)}\right]^{3/2}} \frac{\sinh^2(z)\cos^2(\alpha)(\sinh^2(z) + \sin^2(\alpha))}{\cosh^2(z)}\right) \tag{100}$$

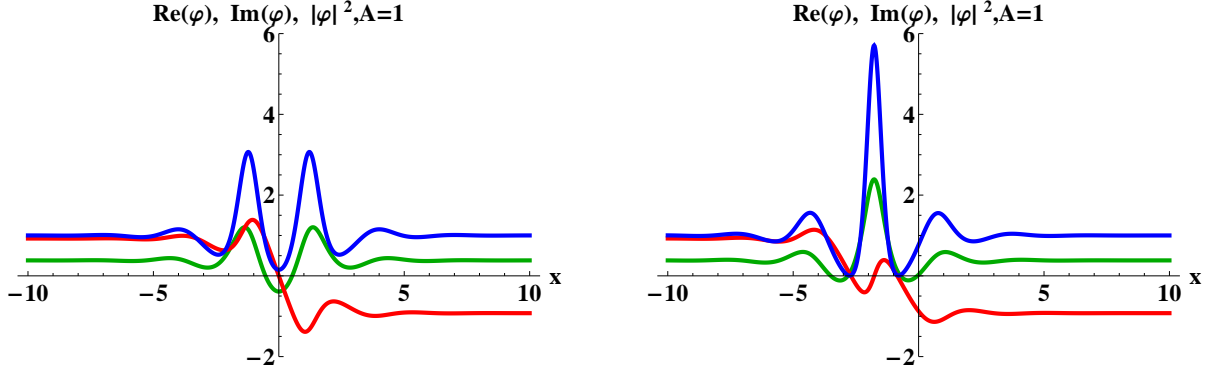The case from Fig. 8 averaged by time is presented on Fig. 9 It is interesting to perform averaging of

16

Figure 8: General solution at the moment of minimum(left) and maximum(right) of its amplitude. $R = 2, \alpha = \frac{5\pi}{16}$. $Re(\varphi)$ - Green, $Im(\varphi)$ - Red, $|\varphi|^2$ - Blue
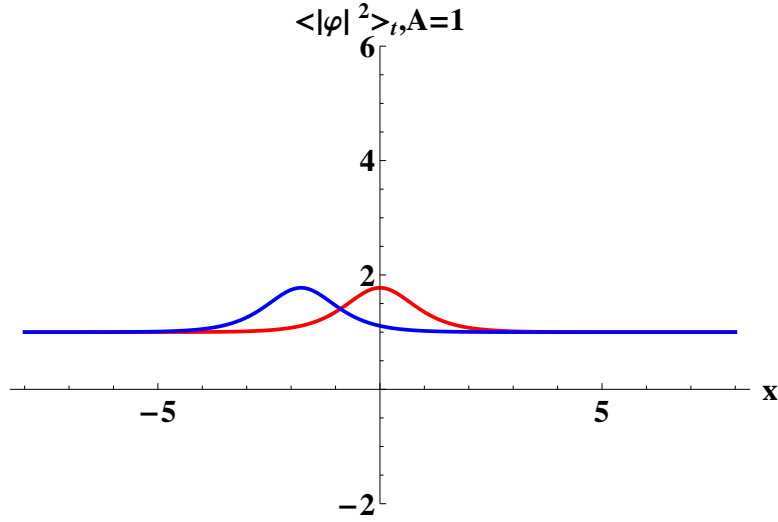

Figure 9: $R = 2, \alpha = \frac{5\pi}{16}$

Kuznetsov-Ma solution in the point $x = 0$. In this case $\alpha = 0$, $u = 0$ and

$$< |\varphi|^2 >_T = A^2 \left(1 + \frac{\sinh^4(z)}{\cosh^2(z)} \right) \tag{101}$$

In particulary in the limiting case we obtain the expected result

$$< |\varphi|^2 >_T = A^2 \tag{102}$$

Now we pay attention to the special case of general solitonic solution when $R \to 1, z \to 0$. This is the "near-Akhmediev"solution of very large size

$$L \approx \frac{1}{2zA\cos(\alpha)} \tag{103}$$

moving with very high group velocity

$$V_{group} \approx \frac{A}{\sin(\alpha)z} \tag{104}$$

17

and very low phase velocity

$$V_{ph} \approx \frac{2Az\cos(2\alpha)}{\sin(\alpha)} \tag{105}$$

The soliton has inner quasiperiodic structure with characteristic scale $l \approx \frac{2\pi}{A\sin(\alpha)}$. If one observe this solution from a fixed point (suppose being at $x = 0$) the total time of this soliton passing is

$$T \approx \frac{L}{V_{group}} = \frac{1}{2A^2\cos(\alpha)\sin(\alpha)} \tag{106}$$

$T \to \infty$ if $\alpha \to 0$. It is interesting that this time does not depend on z. It is important to stress that the "near-Akhmediev" soliton remain after its passing slowly decaying "tails". The "near-Akhmediev" solution is plotted on Fig. 10.
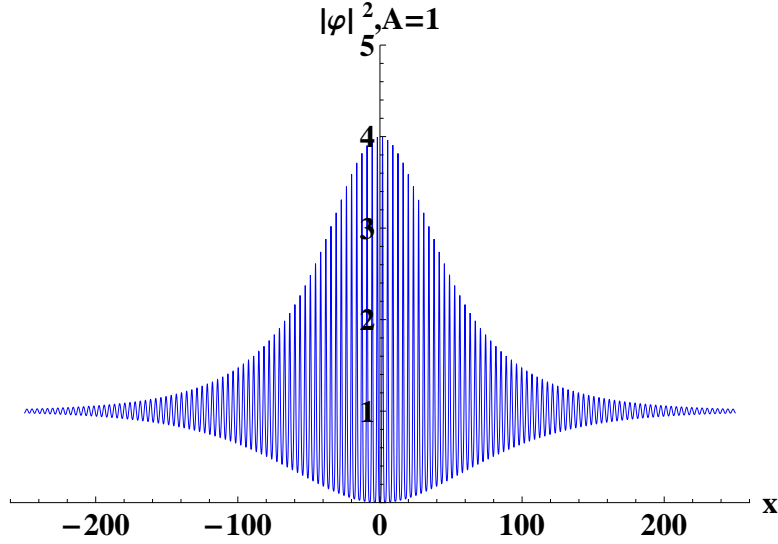

Figure 10: The "near-Akhmediev" solution. ($R = 1.02, \alpha = \frac{\pi}{3}$)

6. Regular two solitonic solution

A general two-solitonic solution on an arbitrary background can be obtained after application of the dressing procedure described in §2 in the relatively simple from.

$$\varphi = A - 2\frac{N}{\Delta} \tag{107}$$

$$N = \frac{B_1}{\lambda_2 + \lambda_2^*} - \frac{B_2}{\lambda_1^* + \lambda_2} - \frac{B_3}{\lambda_2^* + \lambda_1} + \frac{B_4}{\lambda_1 + \lambda_1^*}$$

$$\Delta = \frac{|q_1|^2|q_2|^2}{(\lambda_1 + \lambda_1^*)(\lambda_2 + \lambda_2^*)} - \frac{(\vec{q}_1\vec{q}_2^*)(\vec{q}_1^*\vec{q}_2)}{(\lambda_1^* + \lambda_2)(\lambda_2^* + \lambda_1)}$$

$$B_1 = |q_2|^2 q_{11}^* q_{12} \quad B_2 = (q_1^* q_2)q_{21}^* q_{12} \quad B_3 = (q_1 q_2^*)q_{11}^* q_{22} \quad B_4 = |q_1|^2 q_{21}^* q_{22} \tag{108}$$

Let us denote also function S

$$S = q_{11}q_{22} - q_{12}q_{21} \tag{109}$$

18

One can check the two following identities are valid.

$$B_1 - B_2 - B_3 + B_4 \equiv 0 \tag{110}$$

$$|q_1|^2 |q_2|^2 - (q_1 q_2^*)(q_1 q_2^*) = |q_{11} q_{22} - q_{12} q_{21}|^2 = |S|^2 \tag{111}$$

Thereafter we will study the regular two-solitonic solution such that

$$\alpha_2 = -\alpha_1 = -\alpha \tag{112}$$

Now

$$\lambda_1 = \frac{A}{2}\left(R_1 + \frac{1}{R_1}\right)\cos(\alpha) + \frac{iA}{2}\left(R_1 - \frac{1}{R_1}\right)\sin(\alpha) = A\left[\cosh(z_1)\cos(\alpha) + i\sinh(z_1)\sin(\alpha)\right]$$

$$\lambda_2 = \frac{A}{2}\left(R_2 + \frac{1}{R_2}\right)\cos(\alpha) - \frac{iA}{2}\left(R_2 - \frac{1}{R_2}\right)\sin(\alpha) = A\left[\cosh(z_2)\cos(\alpha) - i\sinh(z_2)\sin(\alpha)\right] \tag{113}$$

To simplify the final expression for two-solitonic solution we perform corresponding shifts in time and space making $|C_1| = |C_2| = 1$. Hence now

$$C_1 = e^{i\theta_1} \qquad C_2 = e^{i\theta_2} \tag{114}$$

Finally the solutions depends on five parameters $R_1, R_2, \alpha, \theta_1, \theta_2$. Let us introduce combination of two last parameters.

$$\theta^+ = \theta_1 + \theta_2 \qquad \theta^- = \theta_1 - \theta_2 \tag{115}$$

Now

$$q_{11} = e^{-\phi_1} + e^{-i\alpha - z_1 + \phi_1} \qquad q_{21} = e^{-\phi_2} + e^{i\alpha - z_2 + \phi_2}$$

$$q_{12} = e^{-i\alpha - z_1 - \phi_1} + e^{\phi_1} \qquad q_{22} = e^{i\alpha - z_2 - \phi_2} + e^{\phi_2} \tag{116}$$

$$\phi_1 = u_1 + iv_1 \qquad \phi_2 = u_2 + iv_2$$

$$u_1 = \text{æ}_1 x - \gamma_1 t \qquad v_1 = k_1 x - \omega_1 t - \frac{1}{2}\theta_1$$

$$u_2 = \text{æ}_2 x - \gamma_2 t \qquad v_2 = k_2 x - \omega_2 t - \frac{1}{2}\theta_2$$

$$\text{æ}_1 = \frac{A}{2}\left(R_1 - \frac{1}{R_1}\right)\cos(\alpha) = A\sinh(z_1)\cos(\alpha)$$

$$k_1 = \frac{A}{2}\left(R_1 + \frac{1}{R_1}\right)\sin(\alpha) = A\cosh(z_1)\sin(\alpha)$$

$$\gamma_1 = -\frac{A^2}{4}\left(R_1^2 + \frac{1}{R_1^2}\right)\sin(2\alpha) = -\frac{A^2}{2}\cosh(2z_1)\sin(2\alpha)$$

$$\omega_1 = \frac{A^2}{4}\left(R_1^2 - \frac{1}{R_1^2}\right)\cos(2\alpha) = \frac{A^2}{2}\sinh(2z_1)\cos(2\alpha)$$

$$\text{æ}_2 = \frac{A}{2}\left(R_2 - \frac{1}{R_2}\right)\cos(\alpha) = A\sinh(z_2)\cos(\alpha)$$

$$k_2 = -\frac{A}{2}\left(R_2 + \frac{1}{R_2}\right)\sin(\alpha) = -A\cosh(z_2)\sin(\alpha)$$

$$\gamma_2 = \frac{A^2}{4}\left(R_2^2 + \frac{1}{R_2^2}\right)\sin(2\alpha) = \frac{A^2}{2}\cosh(2z_2)\sin(2\alpha)$$

$$\omega_2 = \frac{A^2}{4}\left(R_2^2 - \frac{1}{R_2^2}\right)\cos(2\alpha) = \frac{A^2}{2}\sinh(2z_2)\cos(2\alpha) \tag{117}$$

In the special symmetric case

$$\lambda_2 = \lambda_1^*$$
$$\text{æ}_2 = \text{æ}_1 \quad \omega_2 = \omega_1$$
$$k_2 = -k_1 \quad \gamma_2 = -\gamma_1 \tag{118}$$

and solution can be written in an direct analytic form

$$\varphi = A - 2A\frac{M + iK}{H} \tag{119}$$

here

$$
\begin{aligned}
H = \ & 4\big[\cos^2(\alpha) + \sinh^2(z)\big]\big[\cosh^2(z)\sin^2(\alpha)\cosh(4\text{æ}x) - \sinh^2(z)\cos^2(\alpha)\cos(4kx - \theta^-)\big] \\
& + \sinh^2(2z)\cosh(4\gamma t) - \sin^2(2\alpha)\cos(4\omega t + \theta^+) \\
& + 2\sinh(2z)\sin(2\alpha)\sinh(z)\sin(\alpha) \times \\
& \big[\cosh(2\text{æ}x - 2\gamma t)\cos(2kx + 2\omega t + \theta_2) + \cosh(2\text{æ}x + 2\gamma t)\cos(2kx - 2\omega t - \theta_1)\big] \\
& + 2\sinh(2z)\sin(2\alpha)\cosh(z)\cos(\alpha) \times \\
& \big[\sinh(2\text{æ}x - 2\gamma t)\sin(2kx + 2\omega t + \theta_2) + \sinh(2\text{æ}x + 2\gamma t)\sin(2kx - 2\omega t - \theta_1)\big]
\end{aligned}
\tag{120}
$$

$$
\begin{aligned}
M = \ & \sinh(2z)\sin(2\alpha) \times \\
& \Bigg(\sinh(2z)\sin(2\alpha)\big[\cosh(4\gamma t) + \cos(4\omega t + \theta^+)\big] \\
& + 2\sinh(z)\sin(\alpha)\big[\cos^2(\alpha) + \cosh^2(z)\big] \times \\
& \big[\cosh(2\text{æ}x - 2\gamma t)\cos(2kx + 2\omega t + \theta_2) + \cosh(2\text{æ}x + 2\gamma t)\cos(2kx - 2\omega t - \theta_1)\big] \\
& + 2\cosh(z)\cos(\alpha)\big[\sin^2(\alpha) - \sinh^2(z)\big] \times \\
& \big[\sinh(2\text{æ}x - 2\gamma t)\sin(2kx + 2\omega t + \theta_2) + \sinh(2\text{æ}x + 2\gamma t)\sin(2kx - 2\omega t - \theta_1)\big] \Bigg)
\end{aligned}
\tag{121}
$$

$$
\begin{aligned}
K = \ & \sinh(2z)\sin(2\alpha) \times \\
& \Bigg(\sinh(2z)\sin(2\alpha)\sinh(4\gamma t) - \sin(2\alpha)\sin(4\omega t + \theta^+) \\
& - 2\cosh(z)\sin(\alpha)\big[\cos^2(\alpha) + \sinh^2(z)\big] \times \\
& \big[\cosh(2\text{æ}x - 2\gamma t)\sin(2kx + 2\omega t + \theta_2) - \cosh(2\text{æ}x + 2\gamma t)\sin(2kx - 2\omega t - \theta_1)\big] \\
& - 2\sinh(z)\cos(\alpha)\big[\cos^2(\alpha) + \sinh^2(z)\big] \times \\
& \big[\sinh(2\text{æ}x - 2\gamma t)\cos(2kx + 2\omega t + \theta_2) - \sinh(2\text{æ}x + 2\gamma t)\cos(2kx - 2\omega t - \theta_1)\big]
\end{aligned}
\tag{122}
$$

The regular two-solitonic solution in a general non-symmetric case and in the special symmetric case are plotted on Fig. 11 and Fig. 12
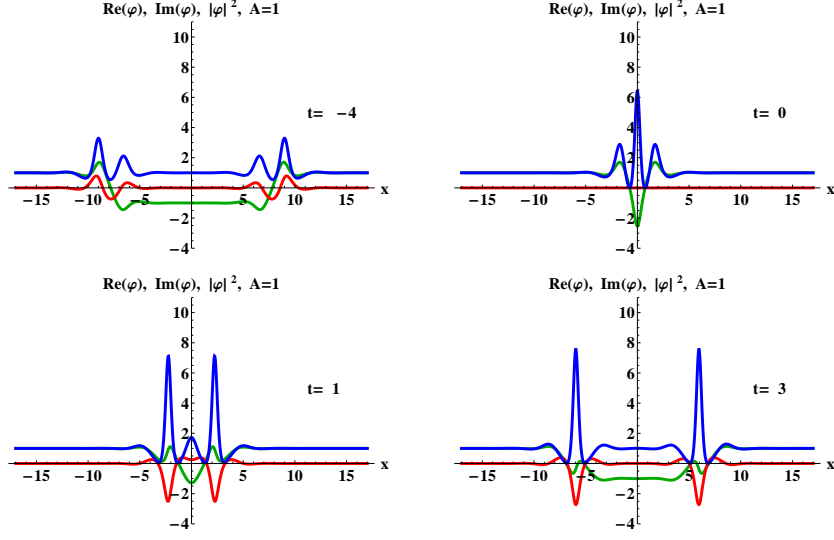
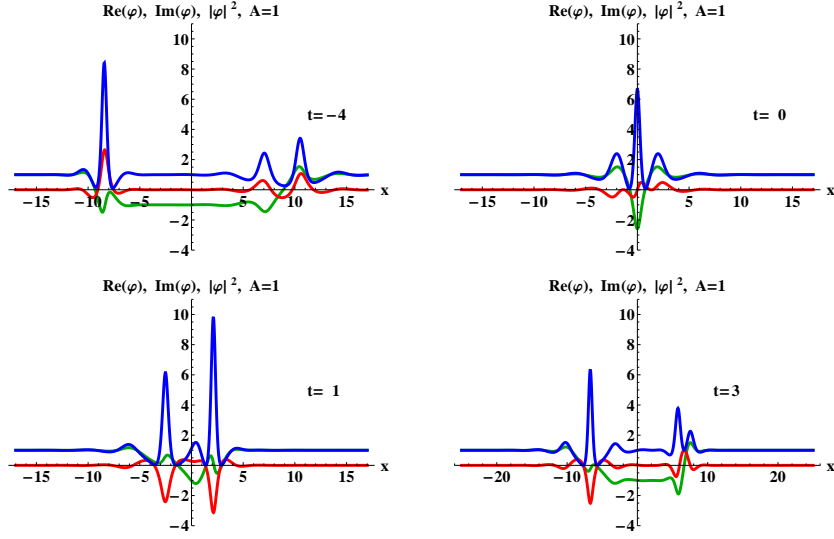Figure 11: Twosolitonic solutions. $R = 2, \alpha = \frac{\pi}{4}$

Figure 12: Twosolitonic solutions. $R_1 = 1.5, \alpha_1 = \frac{\pi}{4}, R_2 = 2.5, \alpha_2 = -\frac{\pi}{4}$

7. Annulation of solitons

Let us consider the limiting case

$$R_1 = R_2 = 1 \tag{123}$$

Now

$$\lambda_1 = \lambda_1^* = \lambda_2 = \lambda_2^* = A\cos(\alpha)$$
$$æ_1 = æ_2 = 0 \qquad \omega_1 = \omega_2 = 0$$
$$k_1 = -k_2 = A\sin(\alpha) \qquad \gamma_1 = -\gamma_2 = \frac{A^2}{2}\sin(2\alpha) \tag{124}$$

21

Looking at (108) we see that now

$$N = \frac{1}{2A\cos(\alpha)}(B_1 - B_2 - B_3 + B_4) = 0 \tag{125}$$

$$u_1 = -u_2 = \frac{A^2}{2}\sin^2(2\alpha)t$$

$$v_1 = A\sin(\alpha)x - \frac{1}{2}\theta_1 \qquad v_2 = -A\sin(\alpha)x - \frac{1}{2}\theta_2 \tag{126}$$

Thus

$$\phi_1 = \phi - \frac{i}{2}\theta_1 \qquad \phi_2 = -\phi - \frac{i}{2}\theta_2 \tag{127}$$

Here

$$\phi = \frac{A^2}{2}\sin(2\alpha)t + iA\sin(\alpha)x \tag{128}$$

Now

$$q_{11} = e^{-\phi+\frac{i}{2}\theta_1} + e^{-i\alpha+\phi-\frac{i}{2}\theta_1} \qquad q_{21} = e^{\phi+\frac{i}{2}\theta_2} + e^{i\alpha-\phi-\frac{i}{2}\theta_2}$$

$$q_{12} = e^{-i\alpha-\phi+\frac{i}{2}\theta_1} + e^{\phi-\frac{i}{2}\theta_1} \qquad q_{22} = e^{i\alpha+\phi+\frac{i}{2}\theta_2} + e^{-\phi-\frac{i}{2}\theta_2} \tag{129}$$

Vectors $\overrightarrow{q}_1, \overrightarrow{q}_2$ are periodic functions on x and exponential functions on time. However simple calculations shows that Δ does not depend on x, t. Now

$$S = q_{11}q_{22} - q_{12}q_{21} = -4\sin(\alpha)\sin\left(\frac{\theta^+}{2}\right) \tag{130}$$

Hence finally

$$\Delta = \frac{\sin^2(\alpha)}{A^2\cos^2(\alpha)}\sin^2\left(\frac{\theta^+}{2}\right) \tag{131}$$

The denominator Δ in this case does not depend on x and t. This is just a number. $\Delta \neq 0$ if $\theta^+ \neq 0$ (we do not consider the special "Peregrine" case $\alpha = 0$). Since $N = 0$ this means that in a general case $\theta^+ \neq 0$ two pure Akhmediev solitons with opposite values of α completely annulate each other. In this case the dressing function χ is presented by the scalar matrix.

$$\chi = \left(1 + \frac{2A\cos(\alpha)}{\lambda - A\cos(\alpha)}\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{132}$$

In the case $\theta^+ = 0$, both numerator N and denominator Δ in (131) are zero. This case must be considerate separately.

8. Small perturbation of condensate

This chapter is the central in our paper. The remarkable fact of exact cancelation of two Akhmediev solitons with opposite values of angular parameter α makes possible to construct special and very important class of $2N$-solitonic solutions describing evolution of infinitesimally small perturbation of condensate. All of these solutions are regular. We start with the case $N = 1$ and assume that R_1, R_2 are close to one (see Fig. 13).

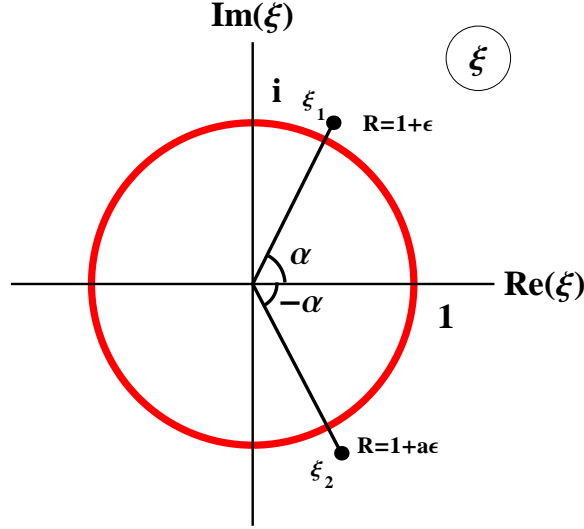$$R_1 \simeq 1 + \epsilon \qquad\qquad R_2 \simeq 1 + a\epsilon \tag{133}$$

Figure 13: Poles location corresponds to small perturbation of condensate

$\epsilon > 0, a > 0, |\epsilon| \ll 1$ Deviation $\delta\varphi$ of a solution from the condensate at the moment of time $t = 0$ is given by expression

$$\delta\varphi = -2\frac{\delta N}{\widetilde{\Delta}} \tag{134}$$

Here δN is a variation of numerator while $\widetilde{\Delta}$ is a "modified" versions of the denominator calculated with higher accuracy. To calculate δN we noticed that

$$\lambda_1 = \frac{A}{2}\left(R_1 + \frac{1}{R_1}\right)\cos(\alpha) + \frac{iA}{2}\left(R_1 - \frac{1}{R_1}\right)\sin(\alpha) \approx A\cos(\alpha) + i\epsilon A\sin(\alpha) \tag{135}$$

$$\lambda_2 = \frac{A}{2}\left(R_2 + \frac{1}{R_2}\right)\cos(\alpha) - \frac{iA}{2}\left(R_2 - \frac{1}{R_2}\right)\sin(\alpha) \approx A\cos(\alpha) - ia\epsilon A\sin(\alpha) \tag{136}$$

Then

$$\lambda_1 + \lambda_2^* = 2A\cos(\alpha) + i(1 + a)\epsilon A\sin(\alpha)$$
$$\lambda_1^* + \lambda_2 = 2A\cos(\alpha) - i(1 + a)\epsilon A\sin(\alpha) \tag{137}$$

Then

$$\delta N = \frac{i(1 + a)\epsilon}{4A\cos^2\alpha}\sin(\alpha)(\widetilde{B}_3 - \widetilde{B}_2) \tag{138}$$

Here $\widetilde{B}_2, \widetilde{B}_3$ are "modified" versions of B_2, B_3. To calculate $\widetilde{\Delta}$ we assume that $\theta^+ \neq 0$ and neglect by deviation of λ_1, λ_2 from its limiting value $A\cos(\alpha)$. Hence as before

$$\widetilde{\Delta} = \frac{1}{4A^2\cos^2(\alpha)}|q_{11}q_{22} - q_{12}q_{21}|^2 = \frac{|S|^2}{4A^2\cos^2(\alpha)} \tag{139}$$

$\widetilde{B}_2$ and $\widetilde{B}_3$ are presented by expressions (108). Vectors $\overrightarrow{q}_1, \overrightarrow{q}_2$ are given by formulae (116), but to calculate "modified" functions we have to keep the first order of small parameter ϵ in functions ϕ_1, ϕ_2. After that functions ϕ_1, ϕ_2 should be considered as not depended on ϵ in the small parameter expansion. Now

$$k_1 = -k_2 = A\sin(\alpha) \tag{140}$$

23

and

$$æ_1 = A\epsilon \cos(\alpha) \qquad æ_2 = aA\epsilon \cos(\alpha) \tag{141}$$

Now

$$\phi_1 = \phi + \epsilon f - \frac{i}{2}\theta_1 \qquad \phi_2 = -\phi + a\epsilon f - \frac{i}{2}\theta_2 \tag{142}$$

Here

$$\phi = iA\sin(\alpha)x \qquad f = A\cos(\alpha)x \tag{143}$$

Vectors $q_1, \quad q_2$ has following components

$$q_{11} = e^{-\phi-\epsilon f+\frac{i}{2}\theta_1} + e^{-i\alpha+\phi+\epsilon f-\frac{i}{2}\theta_1} \qquad q_{21} = e^{\phi-a\epsilon f+\frac{i}{2}\theta_2} + e^{i\alpha-\phi+a\epsilon f-\frac{i}{2}\theta_2}$$
$$q_{12} = e^{-i\alpha-\phi-\epsilon f+\frac{i}{2}\theta_1} + e^{\phi+\epsilon f-\frac{i}{2}\theta_1} \qquad q_{22} = e^{i\alpha+\phi-a\epsilon f+\frac{i}{2}\theta_2} + e^{-\phi+a\epsilon f-\frac{i}{2}\theta_2} \tag{144}$$

Let us denote:

$$\Psi = A\sin(\alpha) \qquad \zeta = A\cos(\alpha) \tag{145}$$

Now

$$S = -4i\sin(\alpha)\left[\cos\left(\frac{\theta^+}{2}\right)\sinh(\epsilon(1+a)\zeta x) - i\sin\left(\frac{\theta^+}{2}\right)\cosh(\epsilon(1+a)\zeta x)\right] \tag{146}$$

Hence

$$\widetilde{\Delta} = \frac{4\sin^2(\alpha)}{A^2\cos^2(\alpha)}\left[\cos^2\left(\frac{\theta^+}{2}\right)\sinh^2(\epsilon(1+a)\zeta x) + \sin^2\left(\frac{\theta^+}{2}\right)\cosh^2(\epsilon(1+a)\zeta x)\right] \tag{147}$$

Then we noticed that

$$\widetilde{B}_3 - \widetilde{B}_2 = (q_{11}^*q_{21} + q_{12}^*q_{22})q_{21}^*q_{12} - (q_{11}q_{21}^* + q_{12}q_{22}^*)q_{11}^*q_{22}$$
$$= q_{22}q_{21}^*(|q_{11}|^2 - |q_{12}|^2) - q_{12}q_{11}^*(|q_{21}|^2 - |q_{22}|^2) \tag{148}$$

Hence

$$\delta\varphi = -2\epsilon Ai(1+a)\sin(\alpha)\left(\frac{\widetilde{B}_3 - \widetilde{B}_2}{|S|^2}\right) \tag{149}$$

Function S are slow oscillating in space. The products $q_{12}q_{22}$ and $q_{21}^*q_{11}^*$ consists both "fast" and "slow" components. Hence φ can be present as a sum of two terms

$$\delta\varphi = \delta\varphi_1 + \delta\varphi_2 \tag{150}$$

Here

$$\delta\varphi_1 = \epsilon Ai(1+a)\frac{\sin(\theta^+)}{\cos^2\left(\frac{\theta^+}{2}\right)\sinh^2(\epsilon(1+a)\zeta x) + \sin^2\left(\frac{\theta^+}{2}\right)\cosh^2(\epsilon(1+a)\zeta x)} \tag{151}$$

is "slow varying in spase function", while

$$\delta\varphi_2 = -\epsilon A(1+a)\frac{1}{\cos^2\left(\frac{\theta^+}{2}\right)\sinh^2(\epsilon(1+a)\zeta x) + \sin^2\left(\frac{\theta^+}{2}\right)\cosh^2(\epsilon(1+a)\zeta x)} \times$$
$$\left(\sin(\alpha)\left[\sinh(2\epsilon a\zeta x)\sin(2\Psi x - \theta_1) + \sinh(2\epsilon\zeta x)\sin(2\Psi x + \theta_2)\right]\right.$$
$$\left. +i\cos(\alpha)\left[\cosh(2\epsilon a\zeta x)\sin(2\Psi x - \theta_1) - \cosh(2\epsilon\zeta x)\sin(2\Psi x + \theta_2)\right]\right) \tag{152}$$

24

is fast oscillating. In symmetric case $a = 1$ and $\delta\varphi$ becomes

$$\delta\varphi = -4\epsilon A \frac{\sin(\alpha)\sinh(2\epsilon\zeta x)\sin\left(2\Psi x - \frac{\theta^-}{2}\right)\cos\left(\frac{\theta^+}{2}\right) - i\cos(\alpha)\cosh(2\epsilon\zeta x)\cos\left(2\Psi x - \frac{\theta^-}{2}\right)\sin\left(\frac{\theta^+}{2}\right)}{\cos^2\left(\frac{\theta^+}{2}\right)\sinh^2(2\epsilon\zeta x) + \sin^2\left(\frac{\theta^+}{2}\right)\cosh^2(2\epsilon\zeta x)} \tag{153}$$

The most simple case appear when $\theta_1 = \theta_2 = \frac{\pi}{2}$ and $a = 1$

$$\delta\varphi = 4i\epsilon A \frac{\cos(\alpha)\cos(2\Psi x)}{\cosh(2\epsilon\zeta x)} \tag{154}$$

The typical small localized perturbation of the condensate at the moment $t = 0$ is presented on Fig. 14.
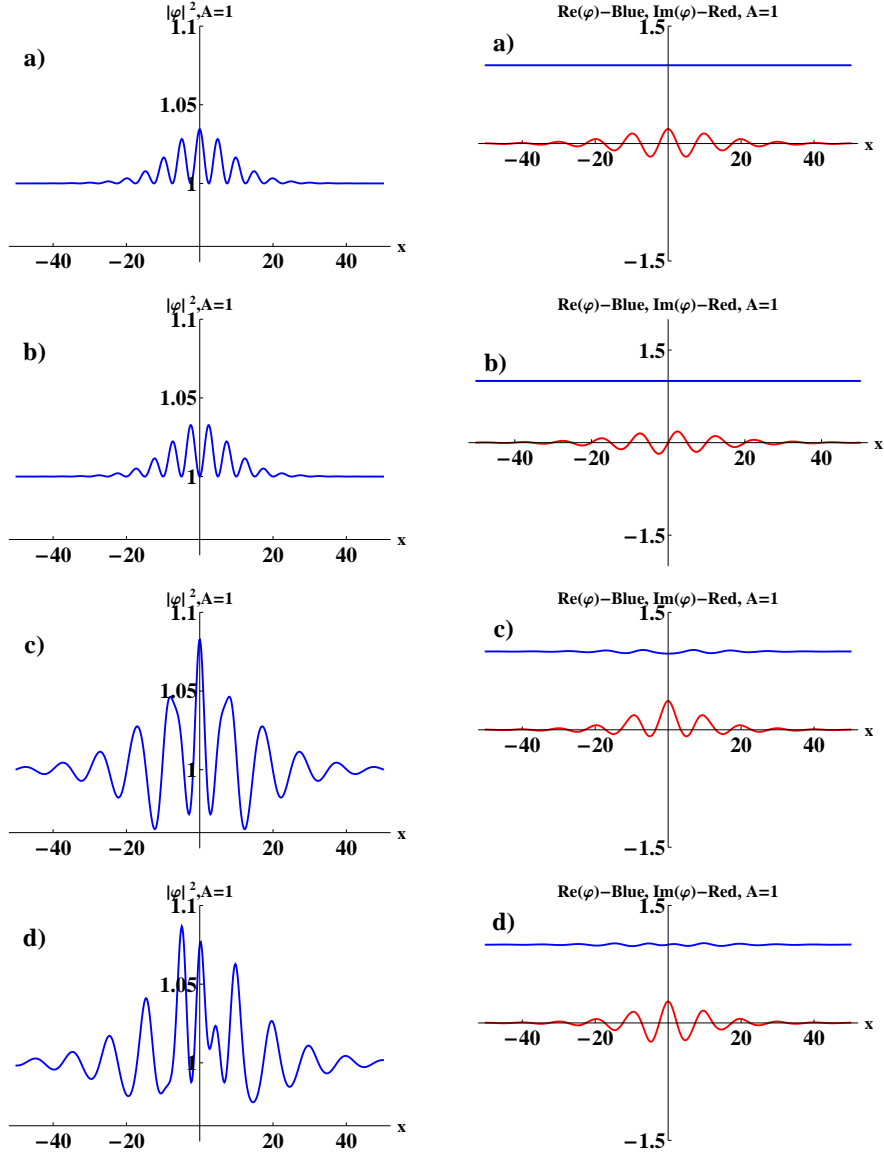
Figure 14: The typical small localized perturbation of the condensate at the moment $t = 0$ a) $a = 1, \epsilon = 0.05, \alpha = \frac{\pi}{10}, \theta_1 = \frac{\pi}{2}, \theta_2 = \frac{\pi}{2}$ b) $a = 1, \epsilon = 0.05, \alpha = \frac{\pi}{10}, \theta_1 = \pi, \theta_2 = 0$ c) $a = 1, \epsilon = 0.05, \alpha = \frac{\pi}{10}, \theta_1 = \pi, \theta_2 = 0$ d) $a = 2, \epsilon = 0.05, \alpha = \frac{\pi}{10}, \theta_1 = \frac{\pi}{2}, \theta_2 = \frac{\pi}{2}$

The comparison of exact solutions and its first order expansion (134) is presented on Fig. 15
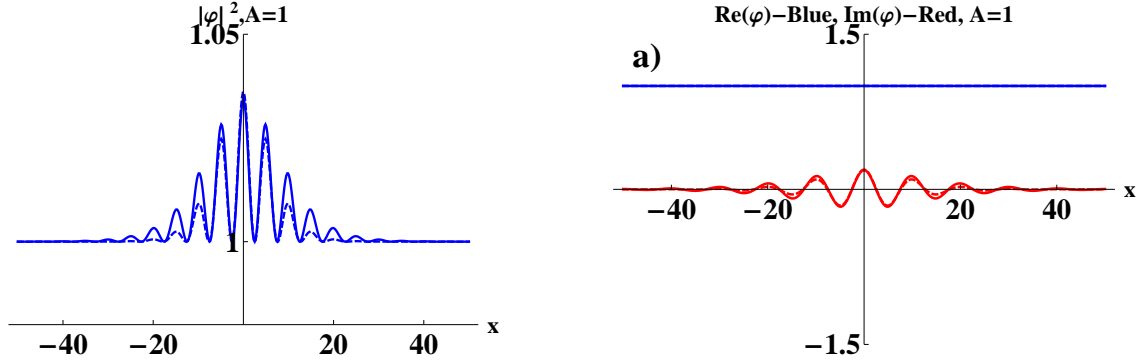
Figure 15: Exact solution (continuous curves), first order expansion (dashed curves) $a = 1, \epsilon = 0.05, \alpha = \frac{\pi}{10}, \theta_1 = \frac{\pi}{2}, \theta_2 = \frac{\pi}{2}$

We end up with the following result. The special class of centered in space and time two-solitonic solutions presenting. At the moment of time $t = 0$ a small perturbation of condensate is described by five parameters $\alpha, \epsilon, a, \theta^+, \theta^-$. Moreover this solution can be shifted in space and in time, thus the total number of free parameters is seven. In construction of such solutions one must be sure that $\theta^+ \neq 0$. The degenerated case $\theta^+ \to 0$ must be studied separately.

The derived 2-solitonic solution describe following physical process.

Initially small localized perturbation of the condensate generates a pair of near-Akhmediev solitons propagating with very fast group velocity in opposite directions. In the symmetric case $a = 1$ these solitons are symmetric. Typical development of these small localized perturbation of the condensate is presented on Fig. 16, Fig. 17, Fig. 18, Fig. 19 In the second part of our paper we will show that this results can be essentially generalized. We construct $2N$ solitonic solutions presenting at $t = 0$ small perturbation of condensate. Also in the second part of paper we will study specially degenerated cases $\theta^+ = 0$
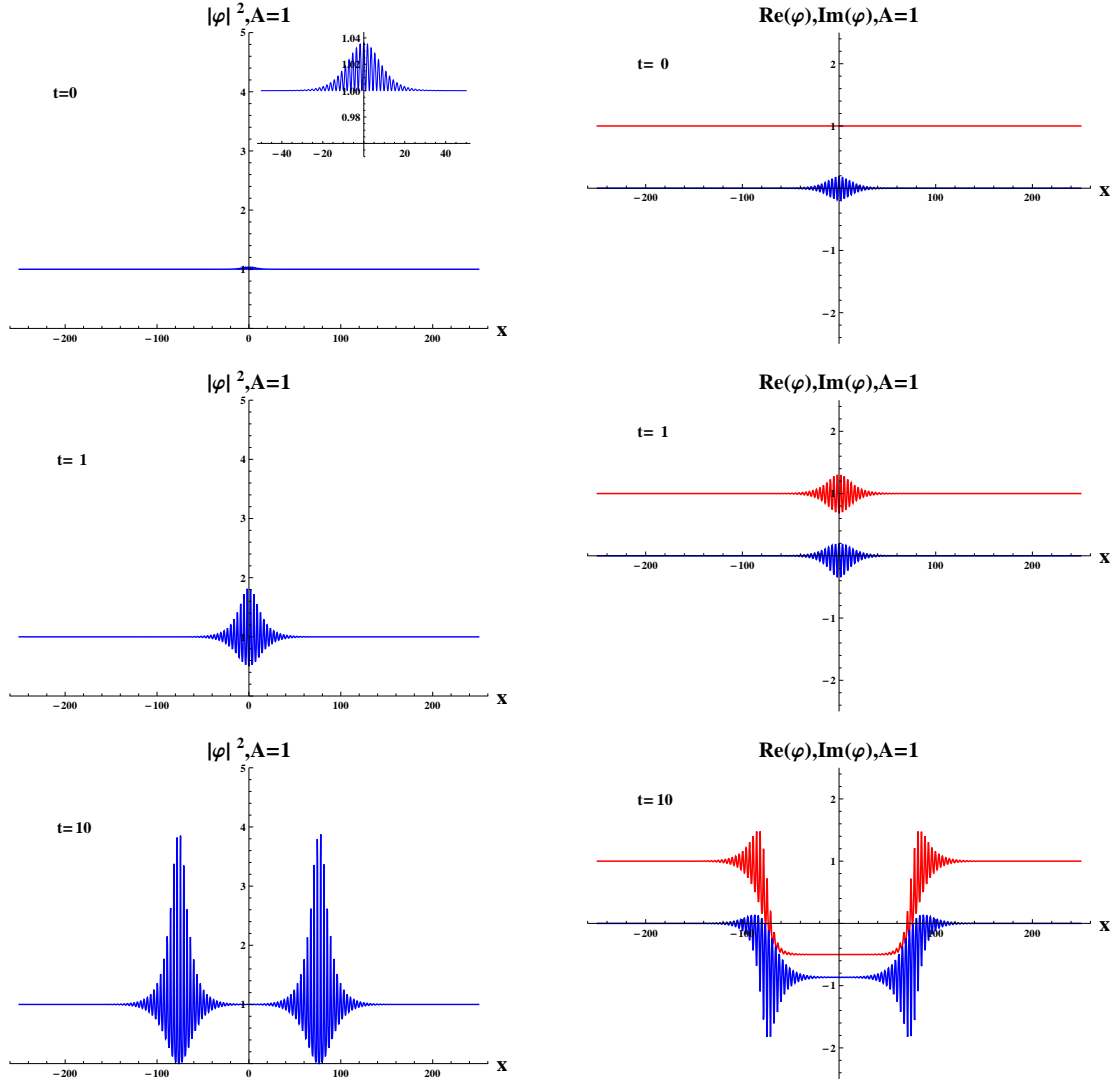
27

Figure 16: Typical development of small localized perturbation of the condensate. $a = 1, \epsilon = 0.2, \alpha = \frac{\pi}{3}, \theta_1 = \frac{\pi}{2}, \theta_2 = \frac{\pi}{2}$
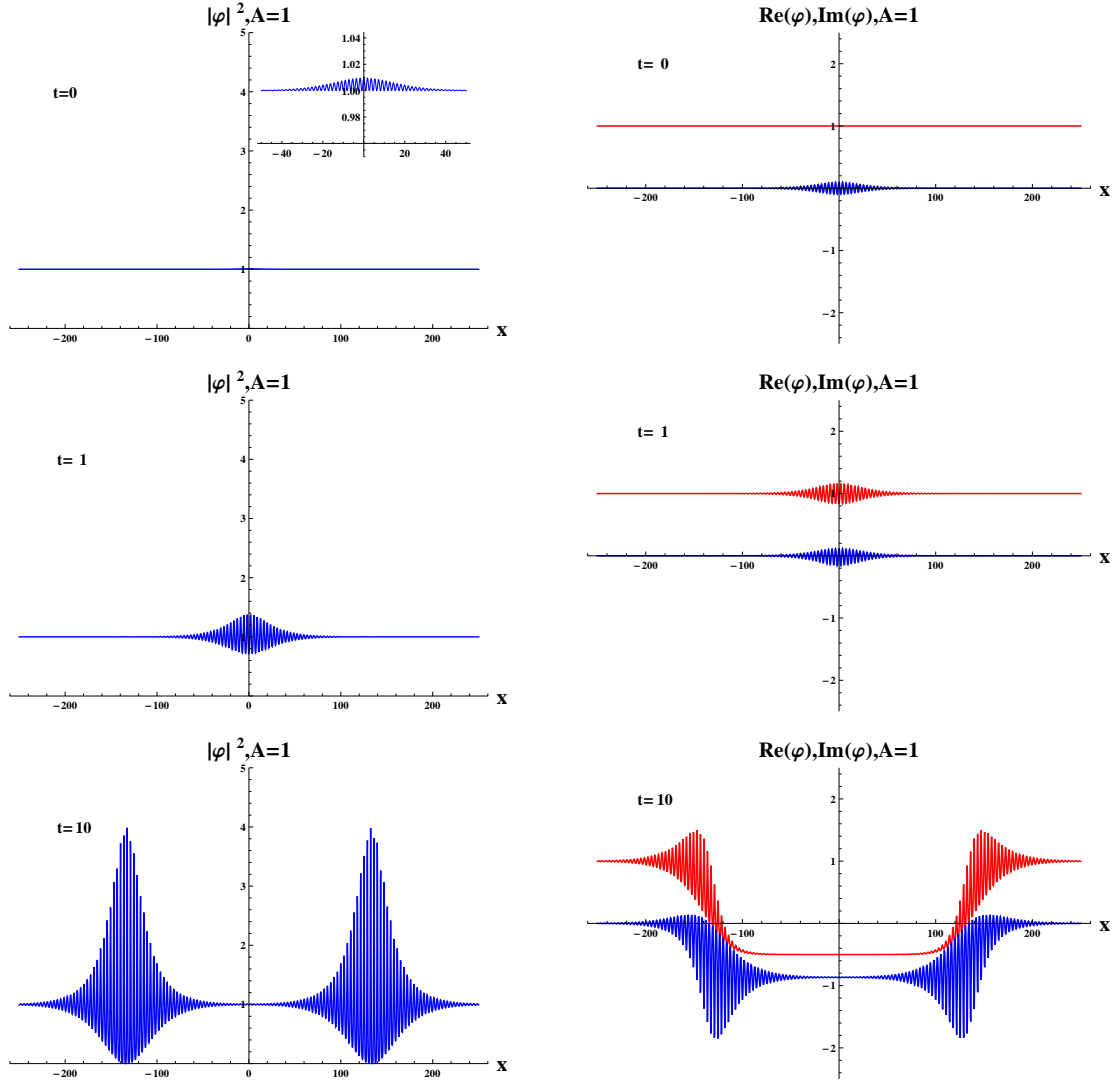
Figure 17: Typical development of small localized perturbation of the condensate. $a = 1, \epsilon = 0.05, \alpha = \frac{\pi}{3}, \theta_1 = \frac{\pi}{2}, \theta_2 = \frac{\pi}{2}$
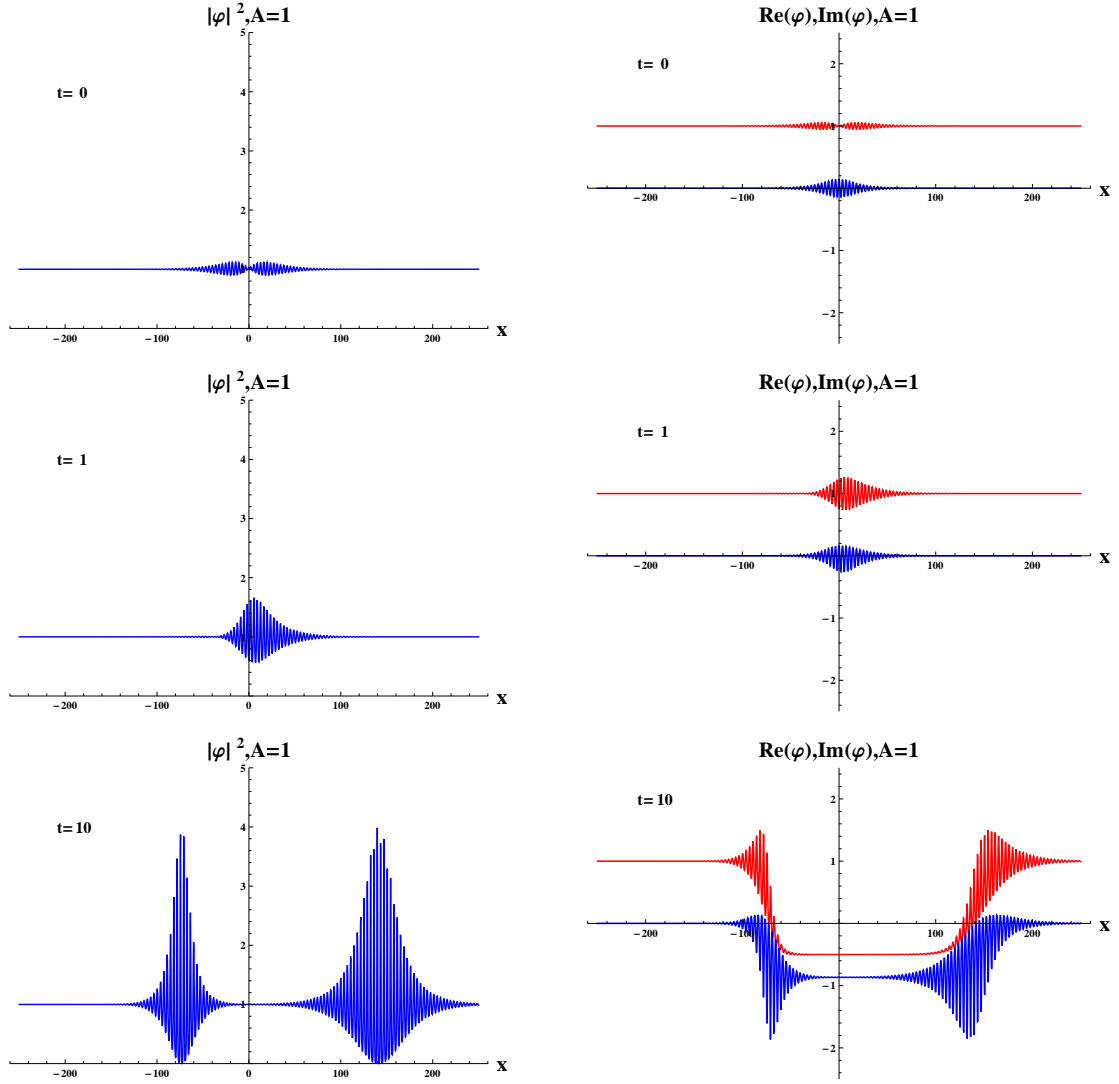
Figure 18: Typical development of small localized perturbation of the condensate. $a = 0.5, \epsilon = 0.2, \alpha = \frac{\pi}{3}, \theta_1 = \frac{\pi}{2}, \theta_2 = \frac{\pi}{2}$
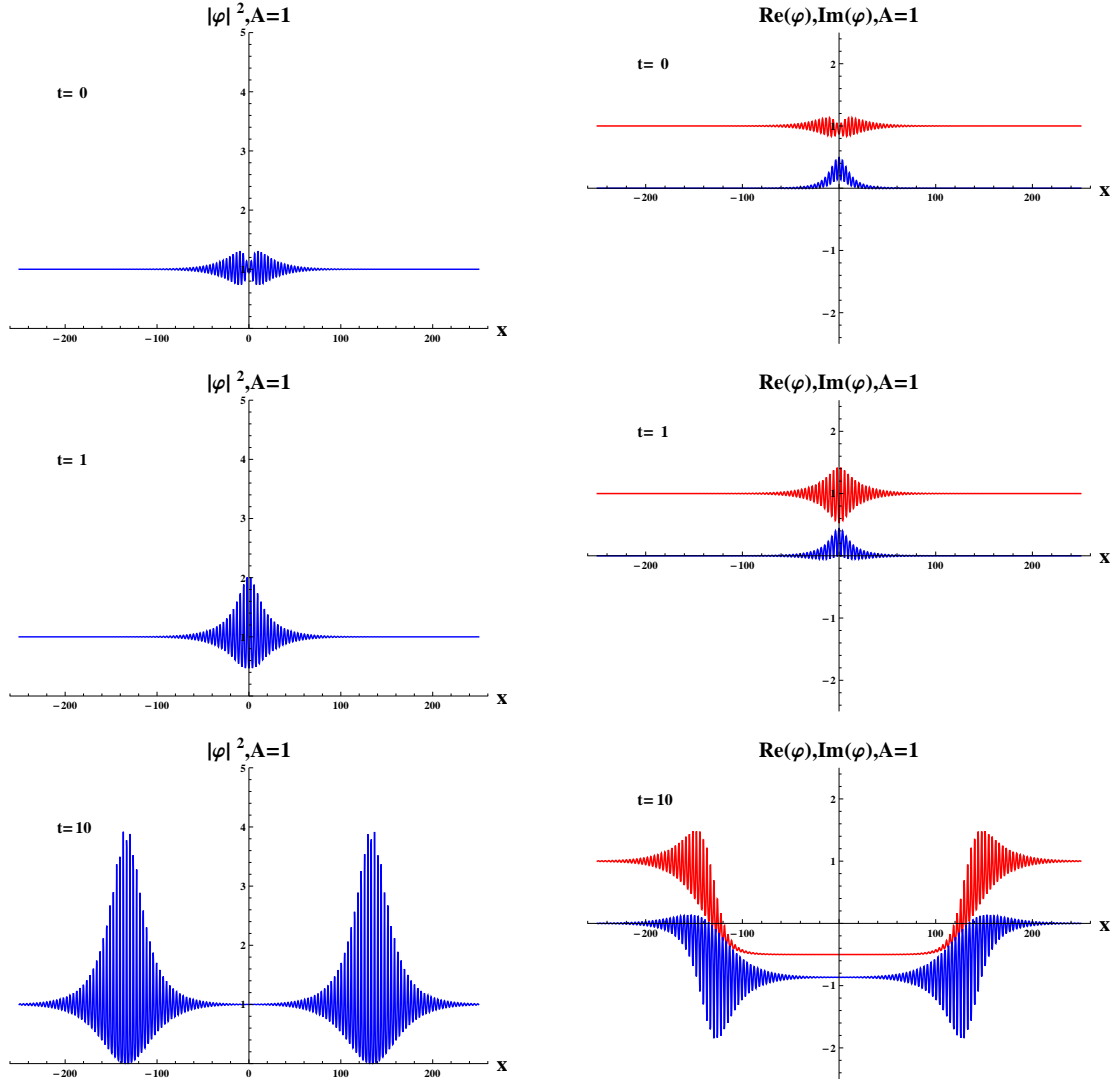
30

Figure 19: Typical development of small localized perturbation of the condensate. $a = 0.5, \epsilon = 0.2, \alpha = \frac{\pi}{3}, \theta_1 = \frac{\pi}{2}, \theta_2 = \frac{\pi}{2}$

9. Acknowledgments

References

[1] V. E. Zakharov, J. Appl. Mech. and Tech. Phys. **9(2)**, 190 (1968)

[2] R. Y. Chiao, E. Garmire and C. H. Townes, Phys. Rev. Lett. **13**, 479 (1964)

[3] A. I. Dyachenko and V. E. Zakharov, JETP Lett. **88**, 356 (2008)

[4] Y. S. Kivshar, G. Agrawal, *Optical Solitons: From Fibers to Photonic Crystals* (Academic Press, 2003)

[5] K. B. Dysthe, K. Trulsen, Phys. Scripta **82**, 48 (1999)

[6] V. Zakharov and A. Dyachenko, Eur. J. Mech. B/Fluids **29**, 127 (2010)

[7] V. E. Zakharov and E. A. Kuznetsov, JETP. **86(5)**, 1035 (1998)

[8] A. A. Balakin, A. G. Litvak, V. A. Mironov and S. A. Skobelev, JETP. **104(3)**, 363 (2007)

[9] V. E. Zakharov, A. I. Dyachenko, A. O. Prokofiev, Eur. J. of Mech. B/Fluids **25(5)**, 677 (2006)

[10] D. Chalikov, D. Sheinin, Journ. Comp. Phys. **210**, 247 (2005)

[11] V. E. Zakharov, A. I. Dyachenko, O. A. Vasilyev, Eur. J. of Mech. B/Fluids **23(1)**, 283-291 (2002)

[12] E. Pelinovsky and C. Harif, *Extreme Ocean Waves* (Springer, 2008)

[13] V. E. Zakharov and A. B. Shabat, Sov. Phys. JETP **34(1)**, 62 (1972)

[14] L. D. Faddeev and L. A. Takhtajan, *Hamiltonian Methods in the Theory of Solitons* (Springer-Verlag, Berlin, Heidelberg, 2007)

[15] C. Sulem and P-L. Sulem, *The nonlinear schrodinger equation. (Self focusing and wave collapse)* (Springer-Verlag, New York, Berlin, Heidelberg ,1999)

[16] C. Kharif, E. Pelinovsky and A. Slunyaev, *Rogue Waves in the Ocean* (Springer ,2009)

[17] E. A. Kuznetsov, *Solitons in a parametrically unstable plasma*, Sov. Phys. - Dokl. (Engl. Transl.) **22**, 507 (1977). *On solitons in parametrically unstable plasma*, Doklady USSR (in Russian) **236**, 575 (1977)

[18] T. Kawata and H. Inoue, J. Phys. Soc. Jpn. **44**, 1722 (1978)

[19] Y.-C. Ma, Stud. Appl. Math. **60**, 43 (1979).

[20] D. H. Peregrine, J. Aust. Math. Soc. Series B **25**, 16 (1983)

[21] B. Kibler, J. Fatome, C. Finot, G. Millot, F. Dias, G. Genty, N. Akhmediev and J.M. Dudley, Nature. Phys. Lett. **6** 790-795 (2010)

[22] A. Chabchoub, N.P. Hoffman and N. Akhmediev, Phys. Rev. Lett. **106** 204502 (2011)

[23] P. Dubard, P. Gaillard, C. Klein, V.B. Matveev, Eur. Phys. J. Special Topics **185**, 247 (2010).

[24] P. Dubard, V.B. Matveev, Nat. Hazards. Earth. Syst. Sci. **11**, 667 (2011)

[25] V. I. Shrira and V. V. Geogjaev, J. Eng. Math. **67**, 11 (2010).

[26] N. Akhmediev, A. Ankiewicz, M. Taki Phys. Lett. A. **373**, 675-678 (2009).

[27] N. N. Akhmediev and V. I. Korneev, Theor. Math. Phys. (USSR) **69**, 1089 (1986).

[28] M. Tajiri and Y. Watanabe, Phys. Rev. E **57**, 3 (1998)

[29] Q.-Han Park and H. J. Shin, Phys. Rev. Lett. **82**, 4432 (1999)

[30] A. Slunyaev, C. Kharif, E. Pelinovsky, T. Talipova Physica D **173**, 7796 (2002)

[31] A. Slunyaev Eur. J. of Mech. B/Fluids **25**, 621635 (2006)

[32] S. L. Lu Li, Zhonghao Li and G. Zhou, Opt. Commun. **234**, 169 (2004).

[33] N. Akhmediev, J. M. Soto-Crespo, A. Ankiewicz Physics Letters A **373**, 21372145 (2009)

[34] N. Akhmediev, J. M. Soto-Crespo and A. Ankiewicz Physical Review A **80**, 043818 (2009)

[35] V. E. Zakharov and A. A. Gelash arXiv:1109.0620 (2011)

[36] V. E. Zakharov and A. B. Shabat Funct. Anal. Appl. **13(3)**, 166 (1979)

[37] V. E. Zakharov, S. V. Manakov, S. P. Novikov and L. P. Pitaevskii, *Theory of solitons. The inverse scattering method* (Plenum Press, New York, London ,1984)

[38] V. E. Zakharov and A. V. Mikhailov Sov. Phys. JETP **47**, 1017, (1978)

[39] A. R. Its, A. V. Rybin and M. A. Sall Theor. Math. Phys. (USSR) **74(1)**, 29 (1988). (in Russian)

[40] D. J. Kedziora, A. Ankiewicz, and N. Akhmediev Physical Review E **85**, 066601 (2012)

[41] V. B. Matveev and M. A. Salli, *Darboux Transformations and Solitons* (Springer Series in Nonlinear Dynamics,Springer-Verlag, Berlin, Heidelberg, 1991)